GEOMETRY

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Published in India by

Takshzila
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www.takshzila.com

MRP: Rs. 350

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Triangles

A polygon (any closed figure with edges being lines) with three sides is called a triangle. Needless to say, a triangle has three vertices, three internal angles formed at these three vertices and three sides.

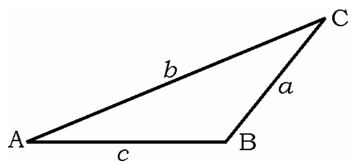
Angle opposite to a side

For any side of a triangle, there are two angles formed at the end points of the sides. The third angle, other than the two angles formed at its ends, is the angle opposite to the side. E.g. In figure, considering side AB, angle opposite to it is angle C.

Side opposite to an angle

For any angle, the rays that form the angle are two sides of the triangle. The third side, other than the two sides forming the angle, is the side opposite to the angle. E.g. In figure, considering angle B, side opposite to it is AC.

Usually, angles are denoted by upper case, e.g. A, B, C and the length of the sides opposite to the respective angles are denoted by lower case, e.g. *a*, *b*, *c*.



Further, one would also know the following types of triangles:

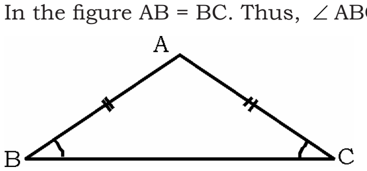
Equilateral Triangle: Triangle in which all the three sides are equal. Also all the three angles will be equal. Thus, measure of each angle will be 60°

Isosceles Triangle: Triangle in which two sides are equal. The angles opposite to the two sides will also be equal.

Note:

In isosceles triangle, if one angle is known, all the three angles can be found out.

In the figure AB = BC. Thus, $\angle ABC = \angle ACB$.



If $\angle A = 80°$, since both angle B and C are equal, say, $x°$, $80 + 2x = 180$ i.e. $\angle B = \angle C = 50°$

If $\angle B = 80°$, angle C will also be 80° and hence $\angle A = 180 - 80 - 80 = 20°$

Thus, in isosceles triangle if one angle is known, the other two will also be known.

Further, if any one angle in an isosceles triangle is 60°, the other two will also be 60° i.e. the triangle will be equilateral.

Scalene Triangle: Triangle in which all three sides are of distinct length.

Angles in a triangle

Sum of all interior angles of a triangle is 180°.

Thus, if two angles of a triangle are known the third can be found out.

E.g. 1: In triangle ABC, measure of angle A is 64°. The internal angle bisectors of angle B and C intersect at D. Find measure of angle BDC.

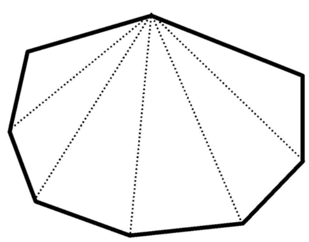
Since A = 64, B + C = 180 – 64 = 116

Also sum of all angles of triangle BDC will be 180 and thus,

$$\angle BDC = 180 - \frac{B}{2} - \frac{C}{2} = 180 - 58 = 122°$$

Sum of interior angles of any polygon

Any polygon having n sides can be broken into $(n-2)$ non-overlapping triangles as shown in the figure.

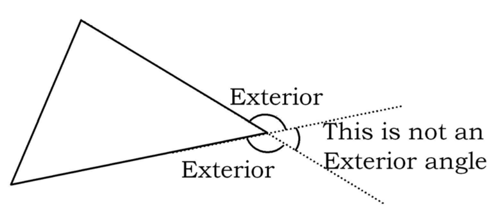


Thus, sum of all interior angles of any polygon with n sides is $(n-2) \times 180°$.

The sum of interior angles of a quadrilateral, pentagon, hexagon and octagon has to be remembered as 360°, 540°, 720° and 1080°

Exterior Angles

The angle formed by any extended side of a triangle with the adjacent side is called an exterior angle.

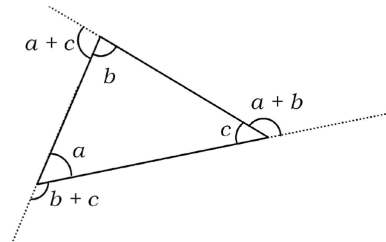


A triangle will have three distinct measures of exterior angles and hence is said to have three exterior angles (and not six).

Measure of exterior angle = Sum of remote interior angles.

The following diagram should make the above relation very obvious:

Since $a + b + c = 180°$ and also a + it's exterior angle = 180°, the angle exterior to angle a has to be equal to $b + c$ i.e. the sum of remote interior angles.

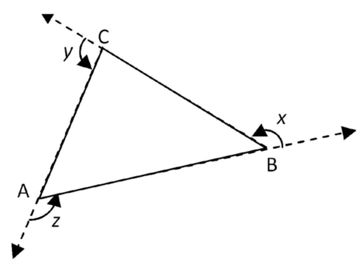


Sum of all exterior angles = 360°

Since each exterior angle is sum of remote interior angles, the sum of all three exterior angles will be $2 \times (a + b + c)$ i.e. $2 \times 180° = 360°$

Meaning of an exterior angle

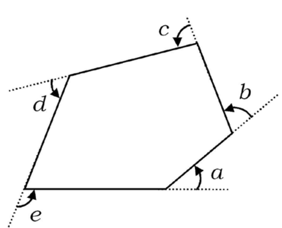
Consider a person who starts from A and travels in the direction AB. At B he changes his direction and starts travelling along BC. The exterior angle formed at B is the angle by which he changed his course. Instead of travelling on straight line AB, he changed his direction by an angle of $x°$.



Again at C he changed his direction by $y°$. On reaching A, say he starts to walk along AB again. Thus at A, he again changed his direction by $z°$.

Since he initially was walking in direction AB and finally is also walking in direction AB, the total degrees by which he must have 'turned' has to be 360°. Thus, sum of exterior angles, $x + y + z = 360°$.

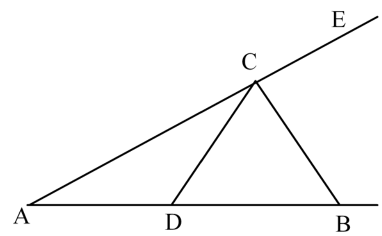
In fact, this is the only aspect common to all convex polygons irrespective of the number of sides. As seen in the following, if the person starts along any side (direction) and after taking 5 turns comes back to the same side (direction), the total degrees by which he must have 'turned' has to be 360.



$a + b + c + d + e = 180°$

Thus for all convex polygons, regular or irregular, sum of all exterior angles = 360°

E.g. 2: In the following figure, if AD = CD = BC, and $\angle BCE = 96^\circ$, what is the measure of $\angle DBC$?



Since AD = CD, $\angle DAC = \angle DCA = x$, say .

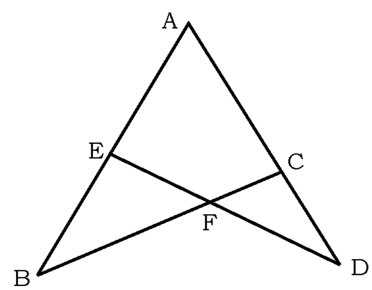
Thus, $\angle CDB = 2x$ because it is the exterior angle of triangle ADC.

Since DC = BC, $\angle CDB = \angle CBD = 2x$.

Next, $\angle BCE$ is the exterior angle of ΔABC and is equal to sum of angles A and B. So, $\angle BCE = 3x = 96$, given . Thus $x = 32$.

$\angle DBC = 2x = 64^\circ$

E.g. 3: In the following figure, if $\angle EFC = 120°$, find the value of $\angle A + \angle B + \angle D$.



Since $\angle EFC = 120°$, $\angle CFD = \angle EFB = 60°$

$\angle ACF = \angle D + 60$ and $\angle AEF = \angle B + 60$, being the exterior angles of triangles CDF and EBF respectively.

In quadrilateral AEFC, sum of all interior angles should add up to 360 degrees, i.e. $\angle A + (\angle B + 60) + 120 + (\angle D + 60) = 360$ i.e. $\angle A + \angle B + \angle D = 120$

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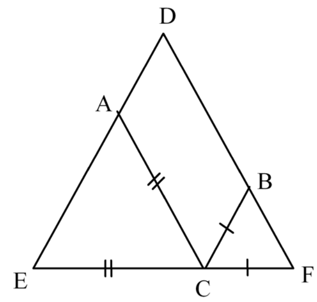
Exercise 1

1. In triangle DEF shown below, points A, B and C are taken on DE, DF and EF respectively such that EC = AC and CF = BC. If angle D = 40 degrees, then what is the measure of $\angle ACB$?



(1) 100 (2) 120 (3) 140 (4) 150

2. Triangle ABC is isosceles with AB = AC. Point P is on BC and point Q on AC such that PA = AQ. If angle BAP = 30, find angle QPC.

(1) 15 (2) 30 (3) 45 (4) 60

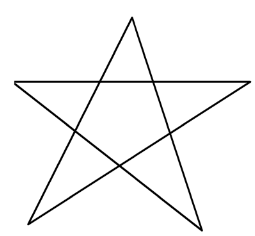
3. In triangle ABC, points D, E and F are on the sides AB, BC and AC respectively such that ED is angle bisector of AEB and EF is angle bisector of AEC. Further angle BDE = angle ADF and angle EFC = angle AFD. Find the measure of angle A.

(1) 15 (2) 30 (3) 45 (4) 60

4. In triangle ABC, D is a point on AB such that AD = AC. If $m\angle ACB - m\angle ABC = 50°$ find the measure of angle DCB.

(1) 50 (2) 25 (3) 35 (4) 15

5. What is the sum of the angles formed at the vertices of a five-pointed star as drawn below:

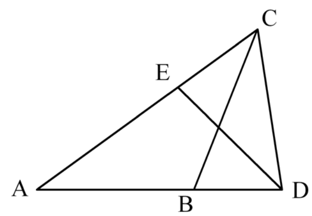


(1) 180 (2) 225 (3) 300 (4) 360

6. ABCD is a square and PAB is an equilateral triangle such that P lies in the interior of the square. Find the measure of $\angle DPC$.

(1) 60 (2) 90 (3) 120 (4) 150

7. In the following figure find the measure of $\angle A$ if AB = BC = CD = DE = EA



(1) 45 (2) 36 (3) 30 (4) 25

8. Points D and E lies on extended sides AB and AC of triangle ABC. The angle bisectors of angle DBC and $\angle ECB$ intersect at F. If $\angle A = 120^\circ$, find the measure of angle BFC.

(1) 30 (2) 60 (3) 90 (4) 150

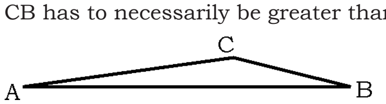
Sides of a triangle

In any triangle, the sum of any two sides has to be greater than the third side.

Thus, if *a*, *b*, *c* are the sides of a triangle, the above property boils down to three conditions viz. $a + b > c$; $a + c > b$; and $b + c > a$.

Rationale

Consider two points, A and B. The shortest distance between the two points has to be the length AB. If one goes from A to B, via any other point C, then the distance travelled AC + CB has to necessarily be greater than the shortest distance AB



In fact this property is sometimes used to prove that three points A, B, C are collinear i.e. lie in a straight line. If of three lengths, AB, BC and AC, sum of any two is equal to the third length, then the three points, A, B and C have to lie along a straight line i.e. have to be collinear.

Thus, the lengths 3, 5 and 9 cannot be the sides of a triangle because 3 + 5 is not greater than 9. It would be a wise thing to always check if the given lengths can form a triangle, before proceeding with any further work.

Corollary

The corollary to this property is that the difference between two sides is always less than the third side.

But one does not need to check both the conditions (sum of two sides being greater than third and difference between two sides being less than third) because if one condition will be violated, the other condition is also violated.

E.g. 4: How many distinct triangles are possible such that two of the sides are 4 and 7 and the third side also has an integral length?

If the unknown side is the largest side, the sum of 4 + 7 has to be greater than the third side. Thus the third side has to be less than 11.

Further, if the third side is the smallest, the sum of it's length and 4 has to be greater than 7. Thus the third side has to be greater than 3.

So the third side could have its length as any natural number from 4 to 10 i.e. 7 different values are possible.

E.g. 5: If the three sides of a triangle are given by $x - 1$, $2x - 4$ and $4x - 12$, find the range of values that x can assume.

The sum of any two sides should be greater than the third side. Thus checking for each pair...

$(x - 1) + (2x - 4) > 4x - 12 \Rightarrow x < 7$

$(x - 1) + (4x - 12) > 2x - 4 \Rightarrow 3x > 9$ i.e. $x > 3$

$(2x - 4) + (4x - 12) > x - 1 \Rightarrow 5x > 15$ i.e. $x > 3$.

Thus the range of values that x can assume is $3 < x < 7$.

Also check that for each of these values the sides are a positive quantity.

The lengths of the sides could also be used to identify if the triangle is an acute angle triangle, right angle triangle or an obtuse angle triangle.

Consider a, b, and c to be the lengths of the three sides of the triangle such that c is the largest side.

Acute angle triangle: Triangle in which all interior angles are acute (measure less than 90°).

For the triangle to be acute, $a^2 + b^2 > c^2$

Right angle triangle: Triangle in which one angle is a right angle (measures 90°).

For the triangle to be right angle triangle, $a^2 + b^2 = c^2$. This is the famous Pythagoras Theorem, as we all would be knowing it.

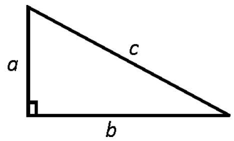
Obtuse angle triangle: Triangle in which one angle is obtuse (measure more than 90°).

For the triangle to be obtuse, $a^2 + b^2 < c^2$

Rather than memorising the above relation, you can logically deduce them as explainend in the following box.

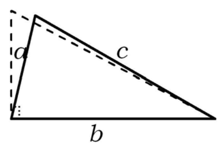
Rationale

Consider a right angle triangle, with c being the hypotenuse i.e. the largest side, and the other two sides to be a and b.

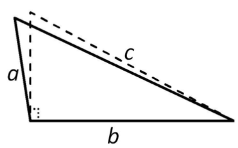


For a right angle triangle the relation $a^2 + b^2 = c^2$ should be well known.

Now if the right angle is reduced a little, so that it becomes acute, and a and b are kept of the same length, it should be obvious that c will reduce i.e. now $a^2 + b^2$ will become greater than c^2.



Alternately if the right angle was increased a little, so that it becomes obtuse, and a and b are kept of the same length, then c has to increase and thus $a^2 + b^2$ will become less than c^2.



E. g. 6: Is a triangle with lengths 10, 16, 12, acute angled, right angled or obtuse angled triangle?

While this question is a straightforward application of what was just learnt, it is given here so that one doesn't make the error of comparing $10^2 + 16^2$ with 12^2. Please note that c in the above explanation has to be the greatest side. Thus, we have to compare $10^2 + 12^2$ with 16^2. Since $10^2 + 12^2 > 16^2$, hence the triangle is acute angled triangle.

E.g. 7: In an isosceles triangle, the unequal side is 10 cms. If the triangle is an obtuse angled triangle, what could be the length of the equal sides assuming they have integral measure?

This question requires you to understand that in an obtuse isosceles triangle, the unequal sides has to be the largest side (else there would be two obtuse angles in the triangle). Thus, if the other two equal sides measure x, then $x^2 + x^2 < 10^2$ i.e. $x^2 < 50$. Thus, the possible integral values for x are 7 and 6. It can't be 5 of lower because then it would not form a triangle (5 + 5 is not greater than 10)

Exercise 2

1. What is the number of distinct triangles with integral valued sides and perimeter 14?

   (1) 3 (2) 4 (3) 5 (4) 6

2. Find the number of distinct acute angles triangles with integral values sides and perimeter being 14.

   (1) 5 (2) 4 (3) 3 (4) 2

3. If the two sides of a triangle are 15 and 24, find the range between which the perimeter of the triangle can lie.

   (1) $9 < p < 39$ (2) $49 \le p \le 77$ (3) $48 \le p \le 78$ (4) $48 < p < 78$

4. The sides of a triangle are given by $8n - 25$, $9n - 48$ and $18n - 91$, where n is a natural number. How many such distinct triangles exist?

   (1) 10 (2) 11 (3) 12 (4) 13

5. Let two sides of a triangle be 1 and 2004. If third side is also an integer, find the perimeter of the triangle.

   (1) 2010 (2) 4008

   (3) 4009 (4) Multiple answers posible

6. The sides of a triangle are $3x + 4y$, $4x + 3y$ and $5x + 5y$ units where $x > 0$, $y > 0$. The triangle is:

   (1) Right-angled (2) Obtuse-angled (3) Equilateral (4) Cannot be determined

7. An isosceles triangle is such that the length of its sides is an integral value. If the two equal sides measure 12 units, what fraction of all such possible triangles are obtuse angled triangles?

   (1) 7/23 (2) 16/23 (3) 1/3 (4) 2/3

8. The sides of a triangle have lengths 11, 15 and k, where k is an integer. For how many values of k is the triangle obtuse?

   (1) 11 (2) 13 (3) 16 (4) Infinite

Relation between sides and angles

In any triangle, side opposite to largest angle is the largest and side opposite to the smallest angle is the smallest. Even the converse of the above is true.

Thus, in triangle ABC if $\angle A = 75°$, $\angle B = 50°$, one can find that $\angle C = 55°$. BC will be the largest side and AC will be the shortest side.

Conversely, if in triangle ABC, AB = 10, BC = 6 and AC = 12, then $\angle B$ will be the largest angle and $\angle A$ will be the smallest angle.

Sine Rule

Since larger angle subtends larger side and smaller angle subtends smaller side, does it imply that the lengths of the sides are directly proportional to the measure of the opposite angles?

Are lengths proportional to the angles?

Even if one doesn't know the answer or logic, one can yet check for the above using any known triangle. Thus, if a triangle has angles 30°, 60° and 90°, then for the above to be true, the lengths of the sides opposite respective angles should be in the ratio $k : 2k : 3k$. But then the triangle is a right angle triangle and $k^2 + (2k)^2 \neq (3k)^2$. Thus the sides are not proportional to the measure of the angles.

While there is a direct proportionality relation between the sides and the opposite angles, it is not the measure of the angles but ...

Sides are directly proportional to the sine of the angles opposite them.

Thus, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. This is called the Sine Rule.

Alternately this also implies that $a : b : c \equiv \sin A : \sin B : \sin C$ i.e. ratio of sides is same as the ratio of sine of opposite angles.

Use of Sine rule

Sine rule can be used to relate two sides and the two opposite angles, $\frac{a}{\sin A} = \frac{b}{\sin B}$. If any three of them is given, the fourth can be found out.

Further to use the rule, one should also know the values of sine of usual angles ...

θ	30°	45°	60°	90°	120°	135°	150°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$

E.g. 8: In triangle ABC, ∠A=75, ∠B=45°. If length AB = $\sqrt{6}$, find length AC.

Using sine rule, $\frac{AB}{\sin C} = \frac{AC}{\sin B}$.

To use this we will first have to find measure of angle C.

∠C = 180 – 75 – 45 = 60°

Thus, $\frac{AC}{1/\sqrt{2}} = \frac{\sqrt{6}}{\sqrt{3}/2} \Rightarrow AC = \frac{1}{\sqrt{2}} \times \sqrt{6} \times \frac{2}{\sqrt{3}} = 2$

Cosine Rule

When three out of the four measures, viz. lengths of two sides and the angles opposite the two sides, are given, the sine rule help us to find the fourth. However in certain cases, it would not be of much use ...

In triangle ABC, if AB = $2\sqrt{3}$, BC = 2 and ∠B = 30°, find the length AC.

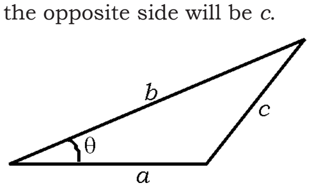
Using sine rule, $\frac{AB}{\sin C} = \frac{BC}{\sin A} = \frac{AC}{\sin B}$ i.e. $\frac{2\sqrt{3}}{\sin C} = \frac{2}{\sin A} = \frac{AC}{\sin 30}$. However this is not of much help because none of the ratio is completely known.

In such cases, specifically when two sides and the included angle is given, we will have to use cosine rule.

In any triangle, $\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$, where θ is the included angle between lengths *a* and *b* i.e. is the angle opposite to length *c*.

Using cosine rule

Start with the angle in mind and then notice that two sides are forming this angle and one side is opposite to the angle. The two sides forming the angles will obviously be *a* and *b* and the opposite side will be *c*.



One would also need to know the value of cosine of usual angles ...

θ	30°	45°	60°	90°	120°	135°	150°
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$

In the above example, since 30° is the included angle between the sides measuring $2\sqrt{3}$ and 2, we have

$$\cos 30 = \frac{(2\sqrt{3})^2 + 2^2 - c^2}{2 \times 2\sqrt{3} \times 2} \Rightarrow \frac{\sqrt{3}}{2} = \frac{16 - c^2}{8\sqrt{3}} \qquad \Rightarrow 12 = 16 - c^2 \Rightarrow c = 2$$

E.g. 9: The three sides of a triangle are 2, $\sqrt{6}$ and $1+\sqrt{3}$. Find the measure of the smallest angle.

The smallest angle will be the angle opposite to the shortest side i.e. opposite to 2 i.e. it will be the included angle between the sides $\sqrt{6}$ and $1+\sqrt{3}$. If this angle is denoted as θ, then

$$\cos\theta = \frac{\left(\sqrt{6}\right)^2 + \left(1+\sqrt{3}\right)^2 - 2^2}{2\times\sqrt{6}\times\left(1+\sqrt{3}\right)} = \frac{6+4+2\sqrt{3}-4}{2\sqrt{6}+2\sqrt{18}}$$

$$\Rightarrow \cos\theta = \frac{6+2\sqrt{3}}{2\sqrt{6}+6\sqrt{2}} = \frac{3+\sqrt{3}}{\sqrt{6}+3\sqrt{2}} = \frac{3+\sqrt{3}}{\sqrt{2}\left(\sqrt{3}+3\right)} = \frac{1}{\sqrt{2}}.$$

Since $\cos 45^\circ = \frac{1}{\sqrt{2}}$, hence the required angle is 45 degrees.

Exercise 3

1. Two angles of a triangle are 30° and 60°. Find the ratio of lengths of the sides of the triangles.

   (1) 1 : 2 : 3 (2) 2 : 3 : 6 (3) $1:\sqrt{3}:2$ (4) Cannot be determined

2. In isosceles triangle ABC, AB = $5\sqrt{3}$ and $\angle$C = 120°. Find length BC.

   (1) 5 (2) 15 (3) $5/\sqrt{3}$ (4) Cannot be determined

3. In triangle ABC, $\angle$A=60°. The angle bisectors of B and C intersect at I. If BI = 5 and CI = 8, find length of side BC.

   (1) $\sqrt{49}$ (2) $\sqrt{89}$ (3) $\sqrt{109}$ (4) $\sqrt{129}$

4. In a triangle ABC, if a = 4, b = 8 and $\angle$C = 60°, then which of the following is necessarily true?

   (1) c = 4 (2) $\angle$B=120° (3) $\angle$A = 30° (4) c = 6

5. ABC is a right angled isosceles triangle with angle B being 90°. If D is a point on AB so that $\angle$DCB = 15° and if AD = 35 cm, then find length CD.

   (1) $35\sqrt{3}$ (2) $35\sqrt{2}$ (3) 105 (4) Cannot be determined

6. In triangle ABC, AB = 6 and AC = 9. AD is the internal angle bisector of angle A and D lies on BC such that BD = 2 and CD = 3. Find the length AD.

   (1) 8 (2) 4 (3) 7.5 (4) $4\sqrt{3}$

Area of a triangle

The area of a triangle can be found using different formulae. Each formula fits appropriately for a particular situation of given data.

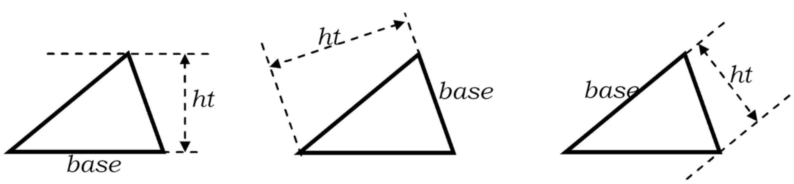
Formula 1: Area = $\frac{1}{2} \times \text{base} \times \text{height}$

Base and Height in a triangle

The base could be any side of the triangle. However if there is a side which is horizontal, it's far more convenient to use it as the base.

Once a side has been chosen as a base, the height is the perpendicular distance from the opposite vertex to this side (or extended side).

Rather than draw the height as a perpendicular line from the opposite vertex to the chosen base, it is strongly recommended to consider the height as the distance been two parallel lines, one line being the side chosen as base and other line is parallel line passing through opposite vertex. This is shown in the below figures for different chosen base.



To use this formula we need to know the height to a particular side of the triangle. And this is typically known only in the case of a right angle triangle. Hence this formula finds use mostly in finding the area of a right angle triangle.

However the formula is very useful in comparing the area of two triangles specifically when the two triangles share a same base or same height, as given in the next topic.

Formula 2: Area = $\sqrt{s(s-a)(s-b)(s-c)}$

This formula is known as Heron's formula. In the formula, a, b, c, refer to the length of the three sides of the triangle and s is the semi-perimeter i.e. $s = \frac{a+b+c}{2}$. Thus the formula is useful in finding the area of a triangle when all the three sides are given.

> Note:
>
> If two sides and the area is given, do not try to use Heron's formula to find the third side. While theoretically it is possible, practically it is almost impossible because we would have to assume the third side to be x and hence s and all the four terms in the root will involve x. Thus, we would get a polynomial of fourth degree inside the root sign.

E.g. 10: In triangle ABC, AB = 35, BC = 24 and AC = 53. Find the length of the altitude AD on side BC.

If the altitude to BC is x, then area of the triangle = $\frac{1}{2} \times x \times \text{BC}$. Also area is given by Heron's formula and thus equating the area found by the two methods, we can find the length of the required altitude.

Thus, $\frac{1}{2} \times x \times 24 = \sqrt{56 \times (56-35) \times (56-24) \times (56-53)}$

$\Rightarrow 12 \times x = \sqrt{56 \times 21 \times 32 \times 3} = \sqrt{8 \times 7 \times 7 \times 3 \times 8 \times 4 \times 3} = 8 \times 7 \times 3 \times 2$

$\Rightarrow x = 28$. Thus, AD = 28.

Formula 3: Area = r × s

In this formula, *r* refers to the in-radius and *s* refers to the semi-perimeter.

We would hardly ever use this formula to find the area of a triangle, but this formula will be used widely as it is one of the most common ways to find the in-radius of a triangle...

E.g. 11: Find the in-radius of the right angle triangle with perpendicular sides being 3 and 4.

Since the triangle is a right angle triangle, its area is $\frac{1}{2} \times 3 \times 4 = 6$.

Also the hypotenuse, third side, can be found using Pythagoras Theorem as $\sqrt{3^2 + 4^2} = \sqrt{9+16} = 5$. Knowing all three sides, we can find semi-perimeter, *s*, as $\frac{3+4+5}{2} = 6$.

Plugging the values found in Area = *r* × *s*, we have 6 = *r* × 6 i.e. *r* = 1.

Thus in-radius of right angle triangle, 3, 4, 5 is 1. Do remember this as we would use this very often later.

The formula can also be used if the triangle is not a right angle, but then finding the area would be a little longer.

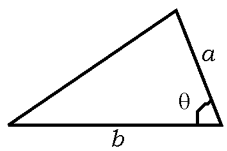
Formula 4: Area = $\frac{1}{2} \times ab\sin\theta$

In this formula, *a* and *b* refer to the length of two sides and θ refers to the included angle between the two sides.

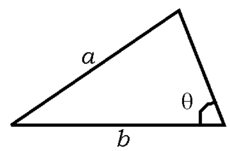
θ has to be the included angle

For those to whom this formula is new, or are scared of trigonometry, there is no trigonometry used here. The only thing to keep in mind is that ...

The angle, θ, whose sine is taken has to be the included angle i.e. has to be the angle formed by the two sides, whose lengths are *a* and *b*. See the diagram



Correct Usage



Incorrect Usage

For your convenience, the values of sine of common angles are reproduced here again ...

θ	30°	45°	60°	90°	120°	135°	150°
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$

While most of us discount this formula, it is very very useful in comparing areas of two triangles if they share a common angle. In doing so, the factor sinθ will get cancelled and we do not need to know any trigonometry to use this formula.

Formula 5: Area = $\dfrac{abc}{4R}$

Here *a*, *b*, *c* refer to the sides of a triangle and *R* refers to the circum-radius (read below) of the triangle. Again the formula would never be used to find the area (if we know, *a*, *b*, *c*, we can use Heron's formula to find the area rather than this), but could be used to find the circum-radius. It has so far never been used in entrance exam scenarios.

There are a few more formula to find the area of a triangle, but then they do not find application in entrance exams and would require knowledge of co-ordinate, determinants or other topics.

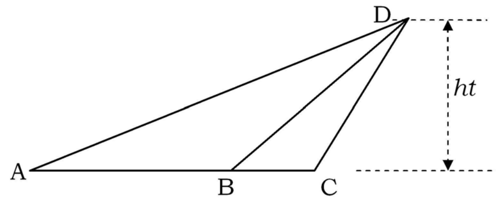
Comparing Areas – Basic Proportionality Theorem

More often than finding the area of a triangle, we would need to compare the areas of two triangles i.e. find the ratio of their areas.

The most common of such questions can be solved easily using the formula that area $= \frac{1}{2}bh$. The ratio of two areas boils down to: $\dfrac{Area_1}{Area_2} = \dfrac{\cancel{1/2} \times b_1 \times h_1}{\cancel{1/2} \times b_1 \times h_1} = \left(\dfrac{b_1}{b_2}\right) \times \left(\dfrac{h_1}{h_2}\right)$.

Thus, to compare the areas, we need to compare the base and the heights. In quite a lot of situations, either the base or the height will be common to the two triangles and hence finding the ratio of the areas becomes an oral question.

E.g. 12: In the following figure, if B divides AC in the ratio 3 : 2, find the ratio of areas of triangles ABD, BCD and ACD.



In the figure, all the triangles, ABD, BCD and ACD have the same height. Thus, the ratio of their areas would be the ratio of their base i.e. the ratio of AB : BC : AC. Since B divides AC in the ratio 3 : 2, the ratio $A(\Delta ABD) : A(\Delta BCD) : A(\Delta ACD)$ would be 3 : 2 : 5.

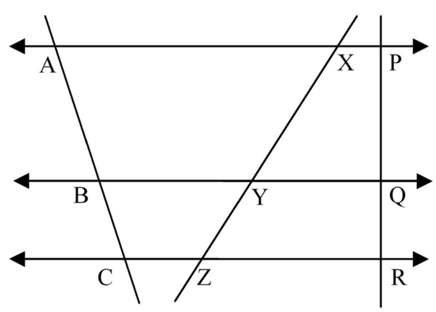
Alternately using Area = $\frac{1}{2} \times ab \sin\theta$

One could also solve it by considering two triangles having a common angle. Thus, we would have to consider triangle ACD and ABD (angle A is common to both). The respective areas would be $\frac{1}{2} \times AD \times AC \times \sin A$ and $\frac{1}{2} \times AD \times AB \times \sin A$ and their ratio would be AC : AB i.e. 5 : 3.

There could be situations where neither the base nor the height is the same but then the ratio of the base and the heights could be found out. We might need to use Basic Proportionality Theorem and hence let's just refresh it.

Basic Proportionality Theorem

With three or more parallel lines, the ratios of intercepts on two or more transversal are equal



If the three horizontal lines are parallel, using BPT, $\frac{AB}{BC} = \frac{XY}{YZ} = \frac{PQ}{QR}$.

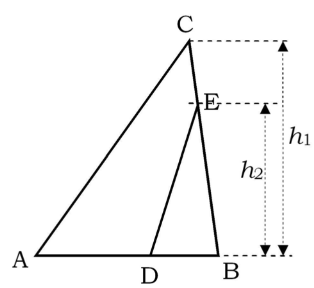
In comparing areas, we would need to find the ratio of the heights. However, the ratio of heights would not be given directly and instead the ratio of an oblique transversal would be given. From this given ratio, the ratio of heights can be found out using BPT. Look at the following example very carefully......

E.g. 13: In triangle ABC, D and E are two points on AB and BC respectively such that AD : DB is 2 : 1 and BE : EC is 3 : 1. Find the ratio of the areas of the triangles ABC and DBE.

Drawing the figure, we get the following figure. Now the required ratio will be $\frac{A(\Delta ABC)}{A(\Delta DBE)} = \left(\frac{AB}{DB}\right) \times \left(\frac{h_1}{h_2}\right)$.

The ratio of the base, $\frac{AB}{DB}$ is given directly to be $\frac{3}{1}$.

The ratio of heights is not given directly, but if we consider the heights as the perpendicular distance between parallel lines, one being the base and other passing through opposite vertex, as shown, it should be obvious.



Using BPT, $\frac{h_1}{h_2} = \frac{BC}{EB}$ i.e. $\frac{4}{3}$

Thus, the required ratio $\frac{A(\Delta ABC)}{A(\Delta BBE)} = \frac{3}{1} \times \frac{4}{3} = \frac{4}{1}$

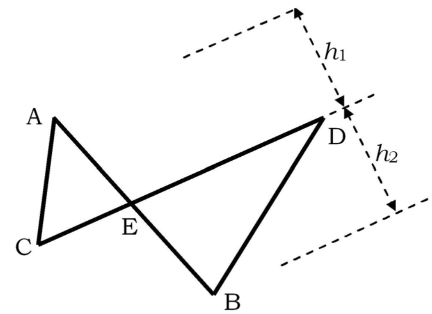
Shortcut

Few students might find using area = $\frac{1}{2} ab \sin\theta$ far more easier than the above.

Considering triangles BDE and BAC and keeping the common angle B in focus, $\frac{A(\Delta BDE) = \frac{1}{2} \times BD \times BE \times \sin B}{A(\Delta BAC) = \frac{1}{2} \times BA \times BC \times \sin B} = \frac{BD}{BA} \times \frac{BE}{BC} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}$

E.g. 14: Two lines AB and CD intersect at E such that AE : EB is 2 : 3 and CE : ED is 3 : 1. Find the ratio of the areas of triangle ACE and BED.

Drawing the figure ...



Considering the base of triangles ACE and BED as CE and ED (along a straight line and also of which the ratio of known), the heights could be thought as the perpendicular distance between the parallel lines, shown as dotted lines in the figure.

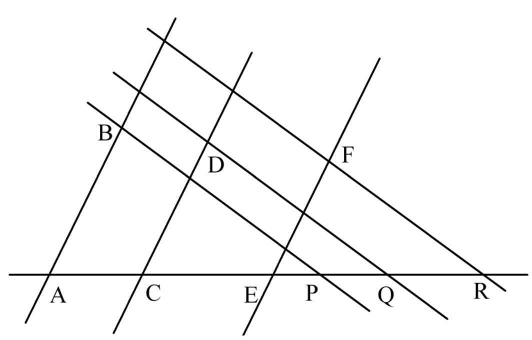
Using BPT it should be obvious that $\frac{h_1}{h_2} = \frac{AE}{EB} = \frac{2}{3}$ and the ratio of the base is given directly, $\frac{CE}{ED} = \frac{3}{1}$. Thus the ratio of the areas is $\frac{2}{3} \times \frac{3}{1} = \frac{2}{1}$.

Shortcut:

Using area = $\frac{1}{2}ab\sin\theta$, $\frac{A(\Delta ACE) = \frac{1}{2} \times AE \times CE \times \sin\theta}{A(\Delta BDE) = \frac{1}{2} \times BE \times ED \times \sin\theta} = \frac{AE}{BE} \times \frac{CE}{ED} = \frac{2}{3} \times \frac{3}{1} = \frac{2}{1}$

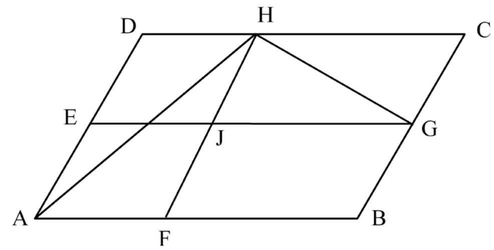
Exercise 4

1. In the figure, AB || CD || EF and BP || DQ || FR such that B, D and F are collinear. If AC = 4, CE = 6, and PQ = 2.4, find the value of QR.



(1) 1.6 (2) 2.4 (3) 3.6 (4) 4.8

2. In the figure below, AD || FH || BC and AB || EG || DC. Point E is the midpoint of AD and H is the point of trisection of DC i.e. ratio DH : DC is 1 : 3. Find the ratio of the areas of the triangles AHF and HJG.



(1) 1 : 1 (2) 1 : 2 (3) 2 : 3 (4) 3 : 4

Directions for Qs 3 & 4: Point D divides side AB of triangle ABC in the ratio 2 : 3 and E divides BC in the ratio 2 : 1.

3. Find the ratio of the area of triangle BDE to the area of triangle ABC.

(1) 2 : 5 (2) 6 : 25 (3) 4 : 25 (4) 4 : 9

4. Find the ratio of the area of triangle DEC to the area of triangle ABC.

(1) 3 : 5 (2) 6 : 25 (3) 1 : 3 (4) 1 : 5

5. In triangle ABC, point D and E are on sides BC and AC such that AD and BE intersect at F. If BF : FE is 3 : 2 and AF : FD is 1 : 3, find the ratio of area of triangles AFE and BFD.

(1) 1 : 3 (2) 1 : 2 (3) 2 : 9 (4) 4 : 9

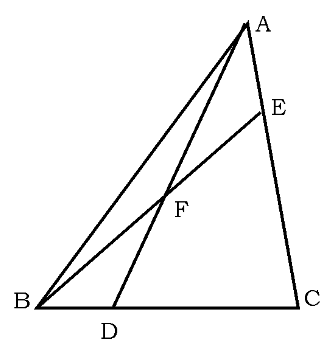
6. Points D, E and F lie on the sides AB, BC and AC of triangle ABC. The ratio AD : DB is 1 : 3, BE : EC is 3 : 2 and CF : FA is 4 : 1. Find the ratio of area of triangle DEF to that of triangle ABC.

(1) 41 : 50 (2) 9 : 50 (3) 83 : 100 (4) 17 : 100

7. The sides AB, BC and CA of triangle ABC are extended to D, E and F respectively such that AB = BD, BC = CE and CA = AF. Find the ratio of area of triangle ABC to that of triangle DEF.

(1) 1 : 5 (2) 1 : 6 (3) 1 : 7 (4) 1 : 8

8. In the figure shown, if the area of triangle ABF is 3 sq. units, that of triangle AFE is 4 sq. units and that of triangle BFD is 2 sq. units, find the area of quadrilateral FECD.



(1) 8 (2) 24 (3) 72 (4) 96

Four Lines and Four Points

In any triangle, for each pair of vertex and opposite side, four specific lines are defined as follows:

Median: Line joining the vertex and the mid-point of the opposite side.

Perpendicular Bisector of a Side: Line perpendicular to the side and bisecting the side as well.

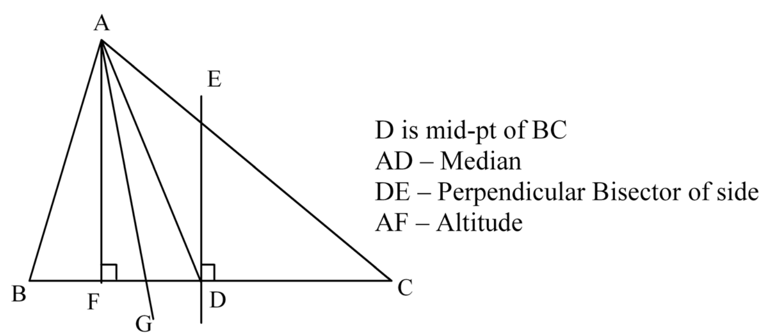
Altitude: Line from the vertex perpendicular to the opposite side.

Confusing?

Do not confuse between the lines:

Line	Passes thru Vertex	Passes thru mid-point of side	Perpendicular to side
Median	Yes	Yes	Not necessary
Perpendicular bisector of side	Not necessary	Yes	Yes
Altitude	Yes	Not necessary	Yes

Angle Bisector: Line bisecting the angle at the vertex.

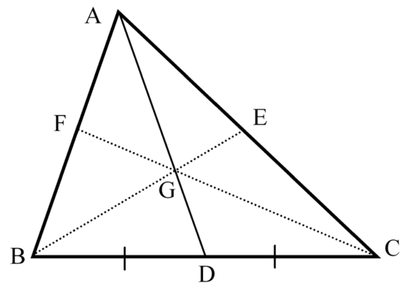


These four lines can be drawn from any of the three vertices. Thus any triangle would have 3 medians, 3 angle bisectors and so on.

For every triangle, the three lines of a particular type are concurrent i.e. they intersect in the same point. And thus four specific points in a triangle are formed. The properties associated with the line and the concurrency point are detailed below and need to be memorized.

Median - Centroid

Centroid is the concurrency point of the medians (point G in the diagram)



D, E & F are midpoints of the sides

Property 1: Each Median divides the triangle into two equal areas i.e. $A(\Delta\text{ABD}) = A(\Delta\text{ADC})$

Property 2: The centroid divides the median in the ratio 2 : 1, with the larger part being towards the vertex.

Thus, AG : GD is 2 : 1. And so is BG : GE and CG : GF.

Property 3: Apollonius Theorem

$AB^2 + AC^2 = 2 \times (AD^2 + BD^2)$

This relation is written with the median AD in mind. The left hand side is the square of the two sides which meet at vertex A. The right hand side has the median and half the third side of the triangle. There could be two more such relations written with BE and CF as the medians in mind.

This is useful in find the length of the median. Thus the formula involves the lengths of the three sides of the triangle and the length of one of the median.

E.g. 15: The length of the sides of a triangle are 21, 13 and 32. Find the length of the median drawn to the longest side.

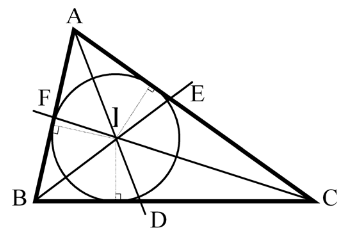
Let x be the length of the median. Since this median is drawn to the longest side, 32, this side will be the opposite side and hence 21 and 13 will be the adjacent sides at the vertex. Thus, using Apollonius Theorem ...

$$21^2 + 13^2 = 2 \times \left(x^2 + \left(\frac{32}{2}\right)^2 \right) \Rightarrow 441 + 169 = 2 \times \left(x^2 + 256\right)$$

i.e. $x^2 = 305 - 256 = 49$ i.e. $x = 7$

Angle Bisector – In-center

In-center is the concurrency point of the angle bisectors (I in the figure)



AD, BE & CF are angle bisectors

The common point on the three angle bisector, I, is at an equal distance from each of the three sides of the triangle i.e. the dotted perpendicular lines are all equal. Hence it is possible to draw a circle which is tangential to all the three sides of the circle.

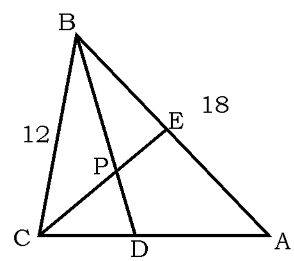
The circle so formed is called an In-circle as it lies 'in' the triangle. And the point I is consequently called In-center.

Property 1: Angle bisector divides opposite side in ratio of adjacent sides i.e.

$$\frac{AB}{AC} = \frac{BD}{DC}$$

E.g. 16: ABC is a scalene triangle with AB = 18, BC = 12 and AC = 15. BD and CE are the angle bisectors with D and E lying on AC and AB. If P is the intersection of the angle bisectors, find the ratio CP : PE

Drawing the diagram,



We require the ratio CP : PE. Since BP is also an angle bisector in smaller triangle BCE, we have $\frac{CP}{PE} = \frac{BC}{BE}$.

BC is given to be 12, thus, to find the required ratio we need to know BE.

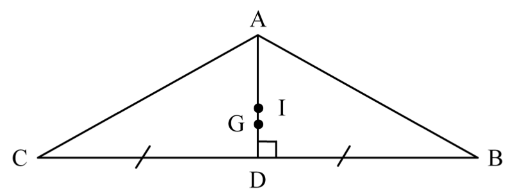
BE can be found using the same property of angle bisector dividing opposite side in ratio of adjacent side, but this time applying it in larger triangle ABC and with CE as angle bisector. BE : EA will be same as the ratio BC : CA i.e. 4 : 5. i.e. E divides BA, 18 cm, in ratio 4 : 5. Thus, BE = 8.

And required ratio, $\frac{CP}{PE} = \frac{BC}{BE} = \frac{12}{8}$ i.e. 3 : 2.

E.g. 17: What is the distance between the in-center and the centroid of an isosceles triangle with length of sides being 17, 17 and 30?

In an isosceles triangle, for the vertex common to the two equal sides, the median, angle bisector, perpendicular bisector of side and altitude are the same line, say line AD in the figure.

In the right triangle ADC, since AC = 17 and DC = 15, we can find AD using Pythagoras theorem as $\sqrt{17^2 - 15^2} = \sqrt{289 - 225} = \sqrt{64} = 8$.



If G is the centroid, we have AG = $\frac{2}{3} \times \text{AD} = \frac{16}{3}$

Since AD is also the angle bisector, I, the in-center, would also lie on it.

In triangle ACD, CI, the angle bisector, divides opposite side, AD, in ratio of adjacent sides, i.e. $\frac{AI}{ID} = \frac{AC}{CD} = \frac{17}{15}$.

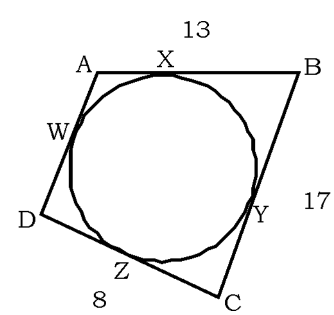
Thus AI = $\frac{17}{32} \times 8 = \frac{17}{4}$. Now, IG can be found as AG – AI = $\frac{16}{3} - \frac{17}{4} = \frac{13}{12}$

Property 2: Angle subtended by any side at the In-center will be 90° plus half the vertex angle.

Three such relations can be written, one of which, with side BC and opposite vertex angle A, is: $\angle \text{BIC} = 90 + \frac{\angle \text{A}}{2}$. We have already proven this in solved example 1.

E.g. 18: Quadrilateral ABCD is such that a circle can be drawn inside the quadrilateral which is tangential to all the four sides of the quadrilateral. If AB = 13, BC = 17, CD = 8, find AD.

Drawing the quadrilateral and naming the tangential points as X, Y, Z, W, we have ...



Assuming AW = AX = x and now moving clockwise,

XB = 13 – x = BY, (because BY and BX are tangents from external point)

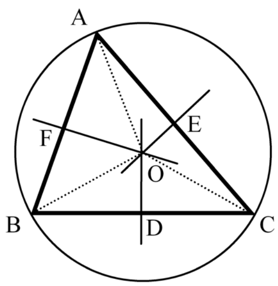
YC = 17 – BY = 17 – (13 – x) = 4 + x = CZ

ZD = 8 – CZ = 8 – (4 + x) = 4 – x = DW

AD = AW + DW = 4 – x + x = 4

Perpendicular Bisector of Side – Circum-center

Circum-center is the concurrency point of the perpendicular bisectors of the side (point O in the figure)



OD, OE & OF are perpendicular bisector of sides

The common point on the three perpendicular bisectors, O, is equidistant from each of A, B and C i.e. OA = OB = OC. Thus, with O as the center, a circle can be drawn and this circle will pass through each of A, B and C.

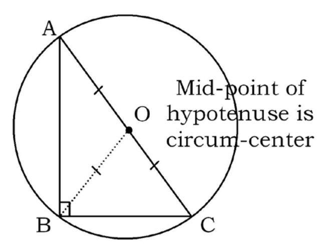
This circle is called Circum-circle, because it circumscribes the triangle. Obviously, it's center, O, will be called the circum-center.

Property 1: In the theory on circles, we will know that the angle at the center of circle is twice the angle formed at the circumference. Hence, $\angle BOC = 2 \times \angle A$.

Two more relations like this can be written for the other two vertex angles and the respective angles formed at the circum-center.

Circum-center of a Right Angled Triangle

Consider the circum-circle of a right angle triangle ...



OA = OB = OC

We know that only angle in a semi-circle is a right angle and ÐABC is a right angle, hence AC has to be the diameter of the circle and its mid-point has to be the center of the circle.

Thus, in a right angle triangle ...

1. The circum-center is the mid-point of the hypotenuse
2. The circum-radius is half the length of the hypotenuse
3. BO will also be the median to the hypotenuse i.e. median to the hypotenuse will also be half the length of the hypotenuse.

E.g. 19: In right angle triangle ABC, right angled at B, the angle subtended by AB at the in-center is 110°. If D is the midpoint of the hypotenuse, find measure of $\angle BDC$

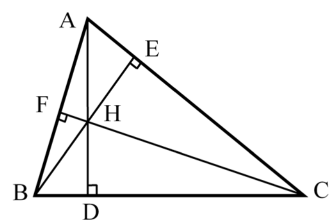
If the in-center is denoted by I, we are given that $\angle AIB = 110^\circ$ and since

$\angle AIB = 90 + \frac{\angle C}{2}$, we have $\angle C = 40^\circ$. And so $\angle A = 50^\circ$.

If D is the mid-point of the hypotenuse, it is also the circum-center and since angle at the circum-center is twice the vertex angle, $\angle BDC = 2 \times \angle A = 100^\circ$.

Altitude - Orthocenter

Orthocenter is the concurrency point of the Altitudes (point H in the figure)



AD, BE & CF are altitudes

Property 1: The angle formed by a side of the triangle at the orthocentre and the vertex angle are supplementary i.e. add up to 180°.

Three such pairs of angles exist, one for each pair of vertex and opposite side. One such relation with vertex A and opposite side BC in mind is $\angle BHC + \angle A = 180^\circ$

Exercise 5

1. In triangle ABC, the internal bisector of the angle A meets BC at D. If AB = $2\sqrt{3}$, AC = $4\sqrt{3}$ and $\angle A=60^\circ$, find the length of AD.

   (1) 3 (2) $3\sqrt{3}$ (3) 4 (4) 6

2. In the parallelogram ABCD, AB = 21, BC = 13 and BD = 14. Find the length AC. (The diagonals of a parallelogram bisect each other)

   (1) 16 (2) 32 (3) 28 (4) 20

3. In triangle ABC, the angle subtended by side BC at the orthocenter is 110 degrees, find the angles subtended by BC at the incenter.

   (1) 120 (2) 125 (3) 140 (4) 70

4. In triangle ABC, measure of $\angle A=60^\circ$. The angle bisectors of B and C intersect at I. If BI = 5 and CI = 8, find length of side BC.

   (1) $\sqrt{105}$ (2) 7 (3) $\sqrt{129}$ (4) $\sqrt{125}$

5. Two of the medians of triangle ABC intersect each other at right angles. If their lengths are 16 and 20 units, find the area of the triangle.

   (1) 640/3 (2) 640/5 (3) 640/7 (4) 640/9

6. In isosceles triangle ABC, AB = AC = 15 units and BC = 18 units. AD, BE and CF are the angle bisector and they intersect at I. Find length AI.

   (1) 8 (2) 4 (3) 15/2 (4) 9/2

7. In triangle ABC, AB = 13 and AC = 21. AD and AE are the median and the altitude drawn from vertex A. Find length of DE if length of median AD is 16.

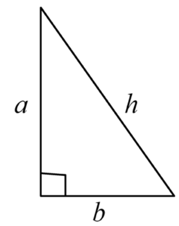
   (1) 68/3 (2) 68/5 (3) 68/7 (4) 68/9

8. The lengths of the medians to the two perpendicular legs of a right angle triangle are 10 and $4\sqrt{10}$. Find the length of the hypotenuse.

   (1) $4\sqrt{3}$ (2) $4\sqrt{5}$ (3) $4\sqrt{11}$ (4) $4\sqrt{13}$

Right Angle Triangles

One would already be aware of Pythagoras Theorem. In a right angle triangle...



$a^2 + b^2 = h^2$

Pythagorean Triplets

Rather than using this theorem in the above sense (and doing the calculation work), one should be familiar with Pythagorean Triplets.

Pythagorean Triplets are lengths of three sides which can form a right angle triangle.

The most popular triplets are:

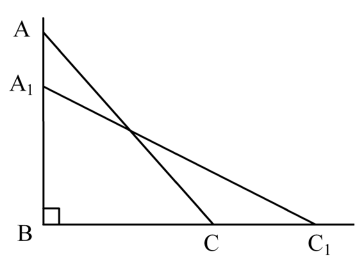
3, 4, 5   5, 12, 13   8, 15, 17   7, 24, 25

If all three numbers of a triplet are multiplied with the same number, it would also result in triplet. Thus few triplets obtained by multiplying {3, 4, 5} are

6, 8, 10   9, 12, 15   12, 16, 20   15, 20, 25

The above triplets and their multiples would take care of more than 90% of the cases of right angle triangles encountered in entrance exams. So mugging them up will save you a lot of calculation effort and time.

E.g. 20: A ladder 65 units long is leaning against a wall with its base at a distance of 25 units from the foot of the wall. If the top of the ladder slips down by 8 units, by what distance does the base of the ladder move away from the foot of the wall?



In the right triangle ABC, the sides are 25, ?, 65.

Since 65 = 5 × 13 and 25 = 5 × 5, we can immediately deduce that the triplet is 5 × {5, 12, 13} i.e. 25, 60, 65.

Thus the top of the ladder was at a height of 60 and after it slipped, it is now at a height of 52.

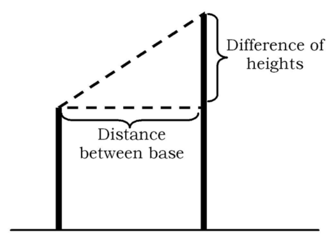
Now, in the right triangle A_1BC_1, the sides are (?, 52, 65).

Since 65 = 5 × 13 and 52 = 4 × 13, we can deduce that the triplet now is 13 × {3, 4, 5} i.e. 39, 52, 65.

Thus the base of the ladder is now 39 units from the foot of the wall .e. it has slipped 14 units away from the foot of the wall.

E.g. 21: Two tall vertical buildings are such that the distance between their top most points is 500 mt and the distance between their base is 140 mts. If the height of one of them is 65 mts, find the height of the other building.

The line joining the top-most points will be the hypotenuse of a right angle triangle with the perpendicular sides being the distance between their base and the difference in their heights …



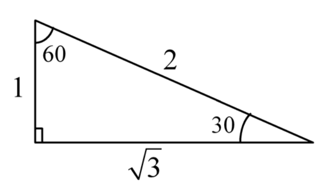
Since the hypotenuse is 500 and one of the perpendicular side is 140, the triplet used can be identified as {7, 24, 25} × 20. Thus the other perpendicular side will be 480.

The difference between the heights of the building is 480 and since one of them is 65 mts tall, the other has to be 65 + 480 = 545 mts.

In right angle triangles, two specific right angle triangles are 30-60-90 and isosceles right angle triangle. Please memorize the following ratio of the sides of these specific triangles.

30-60-90 triangle

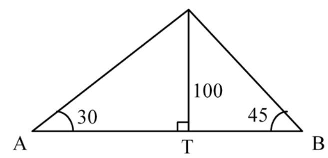
In a 30-60-90 triangle, the ratio of the lengths of the sides opposite to 30, 60 and 90 degrees is $1:\sqrt{3}:2$.



This implies that if one side of a 30-60-90 triangle is known, all the sides can be found out.

Almost all problems based on heights and distances would include a 30-60-90 triangle.

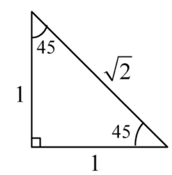
E.g. 22: A and B are two points on either side of the tower such that point A, the base of the tower and point B are in a straight line. The angle of elevation of the top of the tower is 30° and 45° from A and B respectively. If the height of the tower is 100 meters, find the distance between points A and B.



Since side opposite to 30 degrees is 100, side AT, opposite to 60 degrees is $100\sqrt{3}$ and it should be obvious that BT is 100 (being an isosceles triangle). Thus AB = $100\left(1+\sqrt{3}\right)$.

Isosceles Right angle triangle, 45-45-90 triangle

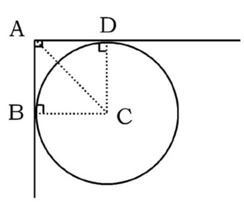
In an isosceles right angle triangle, the ratio of perpendicular sides and the hypotenuse is $1:\sqrt{2}$.



A case of an isosceles right angle triangle is the triangle formed by two adjacent sides of a square and the diagonal. Thus, if the side of a square is a units, its diagonal is $\sqrt{2}a$.

Circle tangential to perpendicular sides

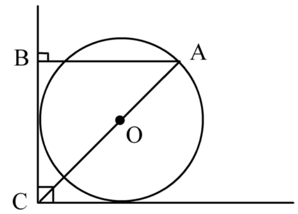
A situation very common is when a circle is tangential to two perpendicular lines, as shown in the figure.



And in all such situations a very very useful approach is to do the following, almost instantaneously …

1. Draw perpendiculars at points of tangency, BC and DC. Their intersection will be the center of the circle
2. ABCD will be a square of side equal to the radius the circle, say *r*.
3. Line AC will form 45° with AD and AB
4. AC = $\sqrt{2}r$, because ABC is a isosceles triangle with perpendicular side being *r*.

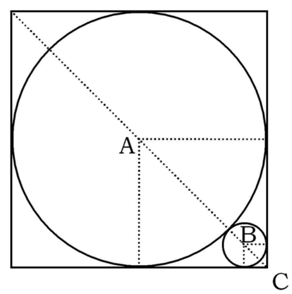
E.g. 23: In the following figure, the line joining the vertex of the right angle, C and the center of the circle O is extended to intersect the circle at A. What is length AB if the radius of the circle is 1 unit.



Since the radius of the circle is 1, CO = $\sqrt{2}$. Thus, AC = $\sqrt{2}+1$.

Further AC will form 45° with CB. Hence, in isosceles right triangle ABC, the hypotenuse AC = $\sqrt{2}+1$ and so the perpendicular side AB = $\frac{\sqrt{2}+1}{\sqrt{2}}$.

E.g. 24: In a square of side 2 unit, a circle is inscribed. Then another circle is inscribe which is tangential to AB and also touches the larger circle, as shown in the figure. Find the radius of the smaller circle.



Drawing the perpendiculars at the point of tangency, we can find the centers of the two circles, say A and B. And both of these will also lie on the diagonal of the square.

The radius of the larger circle is 1 units and let's assume the radius of the smaller circle as *r*.

If the relation between perpendicular sides and hypotenuse of an isosceles triangle was clear, one can immediately conclude that

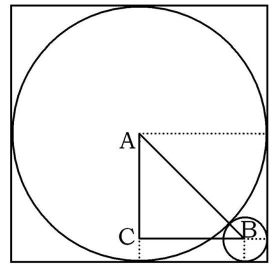
AC = $\sqrt{2}$ and BC = $\sqrt{2}r$. Thus AB = AC – BC = $\sqrt{2}-\sqrt{2}r$.

But since the circle touch each other, AB will also be the sum of the radii i.e. AB = 1 + *r*.

Thus, $1+r=\sqrt{2}-\sqrt{2}r \Rightarrow r=\frac{\sqrt{2}-1}{\sqrt{2}+1}$.

Alternate Method:

Another very useful approach when we have two vertical and two horizontal lines is to work on the difference between their lengths.



Considering the vertical lines, AC will be the difference of the two radii, $1 - r$.

Similarly, BC will also be the difference of the two radii, $1 - r$.

And AB is sum of radii, $1 + r$, since the circle touch each other.

Since ABC is a isosceles triangle, using the ratio of perpendicular sides and hypotenuse, $\sqrt{2} \times (1 - r) = 1 + r \Rightarrow r = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}$.

A thought process so that the above strikes while solving is to think that distance AB is oblique (neither horizontal, nor vertical) and can always be travelled by moving horizontally, BC, and then vertically, CA. Thus AB is the hypotenuse of perpendicular sides BC and CA.

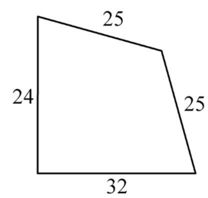
We will see more of right angle triangles in questions based on similarity

Exercise 6

1. From each corners of a square of unit side, right angle triangles are cut off to form a regular octagon. What is the side of the octagon so formed?

(1) 1/3 (2) 2/3 (3) $\frac{1}{2+\sqrt{2}}$ (4) $\frac{\sqrt{2}}{2+\sqrt{2}}$

2. Two sides of a plot measure 32 measure and 24 meters and the angle between them is a perfect right angle. The other two sides measure 25 meters each and the other three angles are not right angles. What is the area of the plot?



(1) 884 (2) 684 (3) 624 (4) 640

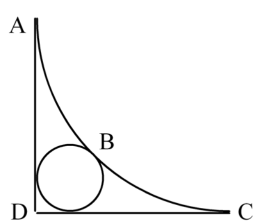
3. 4 equal circles of unit radius are placed such that each circle touches two other circles and the centers of the four circles make a square. Find the radius of a circle which circumscribes all the four circles.

(1) $1+\sqrt{2}$ (2) 2 (3) $4-\sqrt{2}$ (4) $2+\sqrt{2}$

4. A circular table is pushed in the corner of a rectangular room such that it touches the two perpendicular walls of the room. A point on the periphery of the table is such that it is 9 units from one wall and 8 units from the other wall. Find the radius of the table.

(1) 5 (2) 29 (3) 5 or 29 (4) 17

5. In the figure shown, ABC is a quarter of a circle and line AD and CD are tangents to it at A and C respectively. Find the ratio of radius of the circle having ABC as its segment and the radius of the smaller circle drawn (tangential to the lines and the arc) as drawn in the figure.



(1) $4:\sqrt{2}$ (2) $\frac{\sqrt{2}+1}{\sqrt{2}-1}$ (3) 2 : 1 (4) $\sqrt{2}:1$

6. In a square ABCD of unit side, a circle is drawn such that it is tangent to sides AB and BC and passes through vertex D. Find the radius of the circle.

(1) $\frac{\sqrt{2}}{1+\sqrt{2}}$ (2) $\frac{\sqrt{2}-1}{\sqrt{2}}$ (3) $\frac{2}{\sqrt{2}+2}$ (4) $\frac{1}{\sqrt{2}-1}$

7. ABC is a right angle triangle, with angle B being the right angle. Points D and E are on AB and AC respectively such that ADE is an equilateral triangle. If the ratio of the areas of the equilateral triangle and the right angle triangle is 2 : 9, find the ratio of the side of the equilateral triangle and the hypotenuse of the right triangle.

(1) 1 : 2 (2) 1 : 3 (3) 1 : 6 (4) 1 : 4

8. The length of a ladder is exactly equal to the height of the wall. If a ladder is placed on a 2 ft stool, placed 10 ft away from the wall, then the top of the ladder just reaches the top of the wall. Find the height of the wall.

(1) 25 (2) 26 (3) 27 (4) 28

9. The top of a 15 meter high pole makes an angle of elevation of 60° with the top of a sky-scrapper and an angle of depression of 30° with the base of the sky-scrapper. What is the height of the sky-scrapper?

(1) 45 (2) 60 (3) $30\sqrt{3}$ (4) $45\sqrt{3}$

10. Three equal circles touch each other. There are two equilateral triangles formed, one by joining the centers of the three circles and other by three common tangents to each pair of the circle. Find the ratio of the sides of the two equilateral triangles. (Identify the use of 30-60-90 triangle in this case)

(1) $\frac{1}{2+\sqrt{3}}$ (2) $\frac{2}{2+\sqrt{3}}$ (3) $\frac{1}{1+\sqrt{3}}$ (4) $\frac{2}{1+\sqrt{3}}$

Similarity of triangles

Similarity is a very important topic of Geometry and similarity in triangles is extensively tested in CAT and other exams.

Speaking in general terms, two geometrical figures are similar if they are 'similar in shape'.

Speaking in strict mathematic terms, two geometrical figures are similar if the ratios of all the 'corresponding sides' are equal. Corresponding sides are those whose placement/positions correspond to each other in the two figures. We will shortly study a more mathematical meaning for corresponding sides.

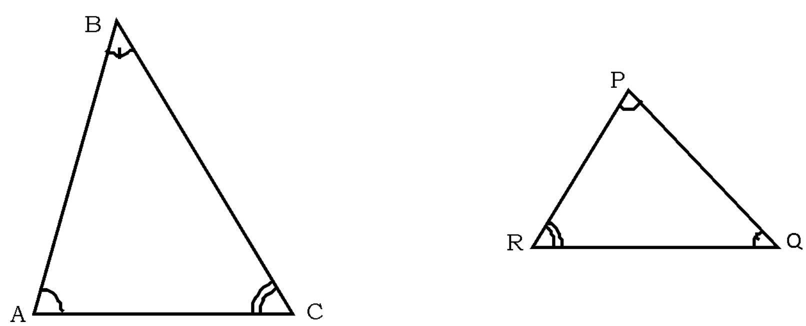
Thus, one can look at similar figures as just a "proportionate" increase/decrease in the size, keeping the shape the same. Thus, lengths of sides of one figure will be the same multiple of the lengths of corresponding sides of the second figure.

Rule for Triangles to be Similar

The rule which will be used to prove that two triangles are similar is the Rule of A-A-A. The rule states that ……

If the measures of the three angles of a triangle are equal to the measures of the angles of another triangle, then the two triangles are similar.

Thus, in the following figure, if $m\angle A = m\angle P$; $m\angle B = m\angle Q$; and $m\angle C = m\angle P$, then the two triangles ABC and PQR are said to be similar.

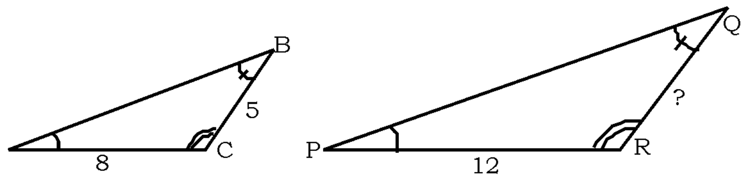


Rule of A – A

Needless to explain, if $m\angle A = m\angle P$ and $m\angle B = m\angle Q$, then it is necessary that the measure of the third pair of angles also has to be equal. Thus, the rule can be shortened to just A – A i.e. if the measures of the two angles of a triangle are equal to the measures of two angles of another triangle, then the two triangles are similar.

Use of Similarity

Once it is proven that two triangles (or any two figures, in general) are similar, invariably the next step involves the use of "ratio of the corresponding sides are equal". Thus, in a general situation, three lengths will be given and the fourth one will be asked ……



Let's assume that the angles are given to be equal, as shown. Thus, the two triangles are similar. Also they are placed such that the similarity (proportionate increase) is obvious.

Let the lengths of the sides be as given and we need to find length of QR.

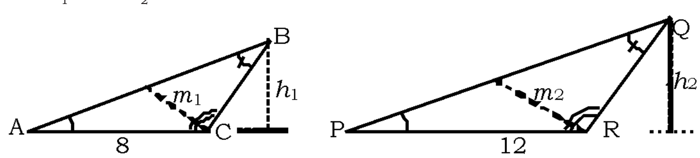
Since they are similar, $\frac{AC}{PR} = \frac{BC}{QR} \Rightarrow \frac{8}{12} = \frac{5}{QR} \Rightarrow QR = 7.5$

It is recommended that one thinks as follows: Taking the two corresponding sides whose lengths are known, AC and PR in this case, we can conclude that the right hand side triangle is 1.5 (=12/8) times the left hand side triangle. Thus, QR will also be 1.5 times BC i.e. 7.5.

Every corresponding 'linear' measure will have same ratio

It is not just the ratio of sides of similar triangles are equal; the ratios of any two corresponding linear measures are the same. (Linear measure is length of any line, which can be measured in meters or cm, unlike area which is measured in m^2)

Thus, if we draw two corresponding heights viz. h_1 and h_2, or two corresponding medians viz. m_1 and m_2 as shown ......



...... then we will also have $\frac{AC}{PR} = \frac{BC}{QR} = \frac{AB}{PQ} = \frac{h_1}{h_2} = \frac{m_1}{m_2} = \frac{\text{in-radius of } \Delta ABC}{\text{in-radius of } \Delta PQR} = ... = \frac{8}{12}$

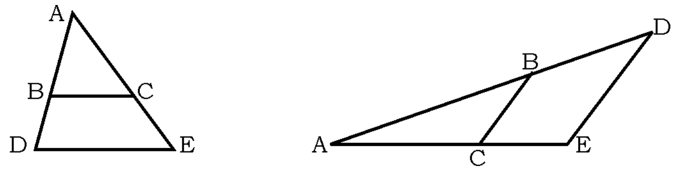
Further, since the ratio of lengths of ΔABC to corresponding lengths of ΔPQR is 2 : 3, the ratio of the area of ΔABC to that of ΔPQR would be square of the ratio i.e. 4 : 9

$$\frac{A(\Delta ABC)}{A(\Delta PQR)} = \frac{\frac{1}{2} \times AC \times h_1}{\frac{1}{2} \times PR \times h_2} = \left(\frac{AC}{PR}\right) \times \left(\frac{h_1}{h_2}\right) = \left(\frac{2}{3}\right)^2$$

Extending the same logic to solids, if two solids are similar with ratio of sides as $a : b$, the surface area of the two solids will be in the ratio $a^2 : b^2$ and the ratio of the volume of the solids will be in ratio $a^3 : b^3$.

Finding Corresponding Sides

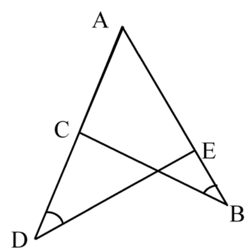
When the two similar triangles are oriented similarly, it is very easy to identify the corresponding sides ……



If BC is parallel to DE, then $\angle ABC = \angle ADE$; $\angle ACB = \angle AED$ and by the rule of A – A, the two triangles, ABC and ADE are similar. The two triangles are oriented identically i.e. ADE is a proportionately increased triangle formed by expanding triangle ABC and there is no 'rotation'. In such cases, the corresponding sides are the sides that are similarly placed i.e. side BC corresponds with side DE; side AB corresponds with side AD (and not with BD, it is not a side of the triangle); side AC corresponds to AE.

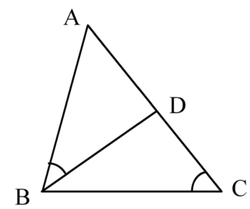
The problem in identifying corresponding sides occurs when the orientation of the triangles are not same. In this case the following approach will help you identify the corresponding sides …

E.g. In the given figure, it is given that $\angle D = \angle B$.



Thus since $\angle A$ is common the two triangles are similar as all angles of one triangle are equal to the angles of other triangle. This is a classic case where one makes mistakes in identify the corresponding sides (in a hurry one usually does $\frac{AC}{AE} = \frac{AD}{AB}$ which is wrong). Considering ΔABC, side opposite to $\angle B$ is AC and in the other triangle, ΔADE, the side opposite the equal angle i.e. $\angle D$ is AE. Thus AC corresponds to AE. Similarly in ΔABC, side opposite to $\angle ACB$ is AB and side opposite to the equal angle i.e. $\angle AED$ is AD. Thus, $\frac{AC}{AE} = \frac{AB}{AD}$. Since in the ratio $\frac{AC}{AE}$, we have taken side of ΔABC in the numerator, in all other ratios we should be taking the side of ΔABC in the numerator.

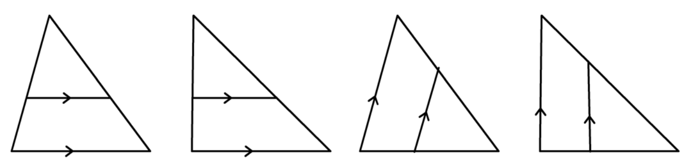
E.g. In the given figure, it is given that $\angle ABD = \angle ACB$.



Again since one pair of angles is given as equal and $\angle A$ is common to the triangles ABD and ACB, the two triangles are equal. Considering smaller triangle ABD as first and triangle ACD as second, the sides opposite the equal angles, $\angle ABD = \angle ACB$ i.e. AD and AB; sides opposite $\angle A$ i.e. BD and BC and the last pair of sides AB and AC are corresponding i.e. $\frac{AD}{AB} = \frac{BD}{BC} = \frac{AB}{AC}$.

Common Similar Figures

The most common figures of similar triangles that need to be immediately identified are as follows:



There are a lot of questions based on similarity of triangles and it would pay off very well for one to get a knack of identifying similar triangles very fast. One simple give-away would be the presence of parallel lines or the presence of a common angle.

In each of the following figures identify pairs of similar triangles (figure (i) has one pair of similar triangles, figure (ii) and (iii) have three pairs of similar triangles. Search for pairs of parallel sides.

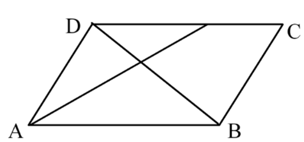


Figure (i)
ABCD is a parallelogram

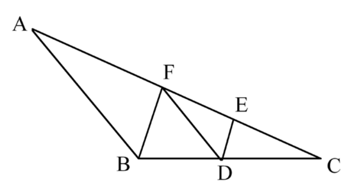


Figure (ii)
AB || DF and BF || DE

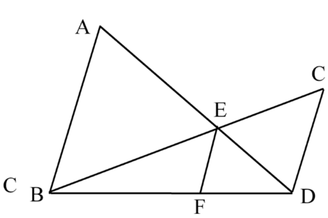
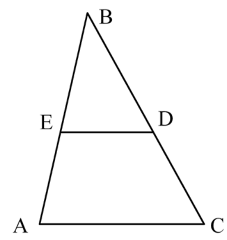


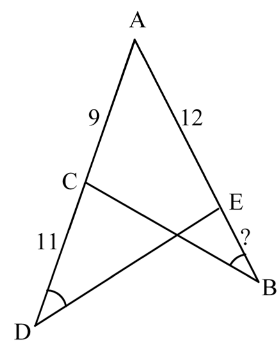
Figure (iii)
AB || EF || CD

E.g. 19: In triangle ABC, D is a point on BC such that BD : DC is 5 : 4. Line DE is parallel to AC such that E is on side AB. Find the ratio of the area of triangle BED to the area of quadrilateral ACDE.



Since DE is parallel to AC, triangles BED and BAC are similar and the ratio of their corresponding sides is 5 : 9. Thus the ratio of their base and also of their height will be the same and so the ratio of their areas will be 25 : 81. Thus the required ratio is 25 : (81 – 25) i.e. 25 : 56.

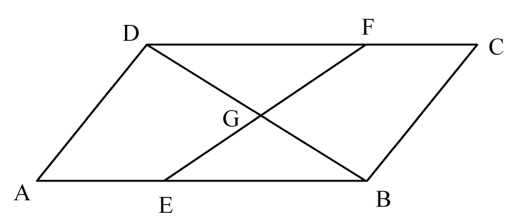
E.g. 20: In the given figure, AC = 9, CD = 11, AE = 12, find length EB, if $\angle D = \angle B$.



In triangles ADE and ABC, $\angle D = \angle B$ and $\angle A$ is common to both the triangles. Thus all three angles of one triangle are equal to the angles of the other triangle. And thus the two triangles are similar. Ratio of the corresponding sides will be equal and thus, $\frac{AC}{AE} = \frac{AB}{AD} \Rightarrow \frac{9}{12} = \frac{AB}{20} \Rightarrow AB = 15$

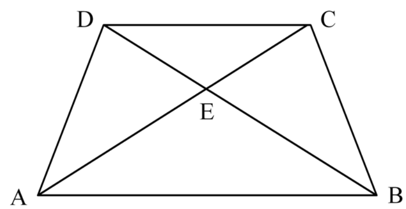
BE = 15 – 12 = 3

E.g. 21: In parallelogram ABCD, point E is on side AB such that AE : EB is 2 : 3 and point F is on side CD such that CF : FD is 3 : 7. Line EF intersects BD at G. Find the ratio of the areas of triangles BEG and DFG.



Triangles BEG and DFG are similar because EB is parallel to DF and thus alternating angles will be equal. The ratio of the corresponding sides will be $\frac{DF}{EB} = \frac{\frac{7}{10} \times DC}{\frac{3}{5} \times AB} = \frac{7}{6}$ since DC = AB. Thus the ratio of the areas of the triangles will be 36 : 42

E.g. 22: Trapezium ABCD has an area of 40 sq. units and the length of parallel sides AB and CD is 12 and 8 respectively. If the diagonals intersect at E, find the sum of areas of ΔAED and ΔBEC.



Triangles AEB and CED are similar because AB is parallel to CD. Further the ratio of the corresponding sides of the two triangles is 12 : 8 i.e. 3 : 2 and hence the height of the two triangles would also be in the ratio of 3 : 2.

Since we know the area of the trapezium and also the lengths of the parallel sides, using the formula, Area = $\frac{1}{2} \times (\text{sum of parallel sides}) \times \text{height}$, we can find the height of the trapezium as 4 units.

So the height of ΔAEB will be $\frac{3}{5} \times 4 = 2.4$ and the height of ΔDEC will be $\frac{2}{5} \times 4 = 1.6$.

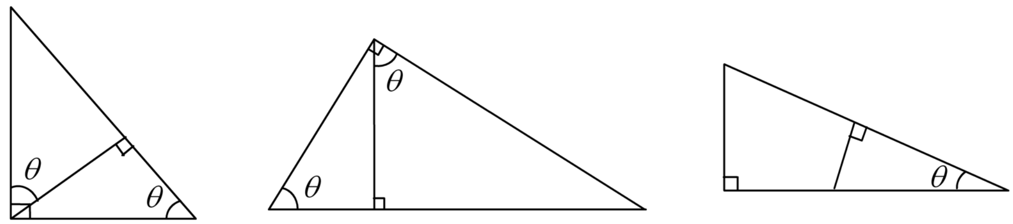
Thus the area of the triangles will be $\frac{1}{2} \times 12 \times 2.4 = 14.4$ and $\frac{1}{2} \times 8 \times 1.6 = 6.4$

(Having found one of the area, the other could also have been found using the fact that the areas would be in ratio of $3^2 : 2^2$ i.e. 9 : 4)

The sum of the areas of ΔAEB and ΔDEC will be 14.4 + 6.4 = 20.8 and thus the sum of the areas of ΔAED and ΔBEC will be 40 – 20.8 = 19.2

Similarity in Right Angle Triangles

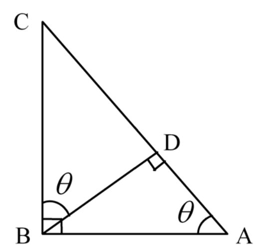
A specific case of use of similarity is in right angle triangles, as shown below.



In figure (i) and (ii) there are three right angle triangles and in figure (iii) there are two right angle triangles. In each figure, all the right triangles are similar as all the triangles have angles equal to θ, 90, 90 – θ.

A very efficient way of solving questions based on above figures is using the funda of similarity. Since all the triangles are similar, if we know the Pythagorean triplet of any one triangle, the other triangle will also be some multiple of the same triplet. See the following examples to build expertise on this approach .....

E.g. 23: In right angle triangle ABC, with $\angle B = 90^\circ$, AB = 6 and BC = 8. D is a point on the hypotenuse AC such that BD is perpendicular to AC. Find lengths of BD, AD and CD.

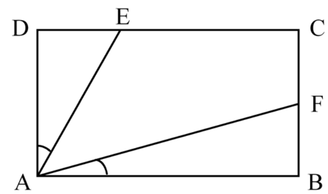


As explained earlier, all the three right angle triangles, ABC, ADB and BDC are similar. And since ABC is based on the triplet (3, 4, 5), so would the other two triangles also be some multiple of (3, 4, 5). Also in the triangle ABC, side opposite to θ is corresponds to a multiple of 4. So in all triangles sides opposite θ would be the side corresponding to 4 out of (3, 4, 5).

In ΔABD, we can assume the sides AD, BD and AB as $3k$, $4k$ and $5k$. Since AB = 6, we can deduce that k = 1.2 and thus AD = 3 × 1.2 = 3.6 and BD = 4 × 1.2 = 4.8. Thus CD = 10 – 3.6 = 6.4.

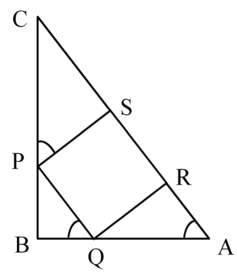
It would be worthwhile to also notice that in ΔBDC, the sides are 4.8, 6.4 and 8 all of which are multiple of the triple 3, 4, 5 (1.6 × (3, 4, 5) = 4.8, 6.4 and 8) as expected.

E.g. 24: In rectangle ABCD with AB = 2 and BC = 1, point E is taken on side CD such that $\angle BAF = \angle EAD$, where F is the mid-point of BC. Find length DE.



It should be obvious that ΔBAF and ΔDAE are similar because each has one right angle and it is given that $\angle BAF = \angle EAD$. Further we know that the ratio of the sides BF and AB of ΔBAF is $\frac{1}{2} : 2 \text{ i.e. } 1:4$. Since the triangles are similar, the ratio of DE and AD would also be 1 : 4. And since AD = 1, ED has to be $\frac{1}{4}$.

E.g. 25: In right angle triangle ABC, AB = 6, BC = 8 and AC = 10. A square PQRS is inscribed in the triangle as shown in the figure. Find the side of the square?



If $\angle A = \theta$, the angles of right angle triangle ABC will be $\theta, 90-\theta, 90$. So would the angles in triangle ARQ.

Since PQRS is a square, PQ is parallel to SR and so $\angle PQB = \theta$. Angle in right triangle QBP would also be $\theta, 90-\theta, 90$. Similarly angles in PSC would also be $\theta, 90-\theta, 90$ as $\angle C = 90-\theta$.

Thus all four right angle triangles seen in the figure are similar and thus are based on the same triplet 3, 4, 5 (as triangle ABC is 2 × (3, 4, 5)). Equal angles are marked in the figure to help find corresponding sides and sides opposite to them correspond to 4 out of the 3, 4, 5.

If BQ is assumed as $3k$, then PQ would be $5k$. Since PQRS is a square QR will also be $5k$.

In right triangle ARQ ratio of sides QR and AQ would be 4 : 5 and since QR is $5k$, so AQ has to be $\frac{5}{4} \times 5k = \frac{25}{4}k$.

$$BQ + AQ = 3k + \frac{25}{4}k = 6 \Rightarrow k = \frac{24}{37}$$

Thus, side of the square, $5k = \frac{120}{37}$.

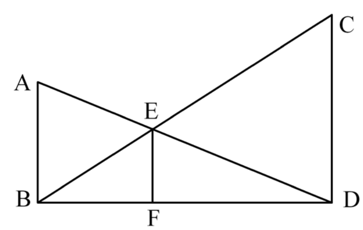
Please learn the above approach very carefully as the solution by any other way is pretty lengthy.

Exercise 7

1. In triangle ABC, a point D is taken on side AB such that $\angle ACD = \angle ABC$. If AC = 12, AD = 6 and CD = 8, find the perimeter of triangle ABC.

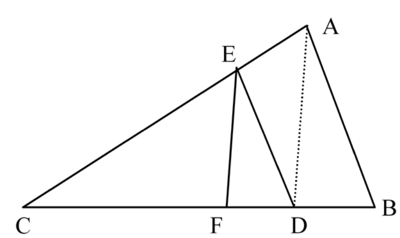
   (1) 52 (2) 48 (3) 56 (4) 42

2. In the figure given, AB, EF and CD are perpendicular to BD. If AB : CD is 3 : 4, find the ratio of BF : FD



   (1) 1 : 1 (2) 3 : 4 (3) 4 : 3 (4) 9 : 16

3. In the given figure, point D divides BC in the ratio 1 : 4. From D a line DE parallel to AB is drawn and from E a line EF parallel to AD is drawn. Find the ratio CF : BC.



   (1) 9 : 25 (2) 3 : 5 (3) 4 : 5 (4) 16 : 25

4. In triangle ABC, line DE and FG are parallel to BC such that D and F lie on side AB and E and G lie on side AC and FG closer to side BC. If the ratio of lengths of AD : DF : FB is 4 : 3 : 2, which of the areas among area of triangle ADE, area of quadrilateral DEGF and area of quadrilateral FGCB is the largest?

   (1) ADE (2) DEGF (3) FGCB (4) DEGF and FGCB

5. In trapezium ABCD with AB and CD being the parallel lines, EF is a line parallel to the parallel sides and it divides the trapezium into two equal areas. If AB = 8 and CD = 12, find the length of EF.

   (1) 10 (2) $\sqrt{96}$ (3) $\sqrt{104}$ (4) 9.6

6. In parallelogram ABCD, point P divides AD in the ratio 1 : 3 and point Q divides BC in the ratio 3 : 1. The diagonal AC intersects lines PB and DQ at E and F respectively. Find the ratio of AE : EF : FC.

   (1) 1 : 3 : 1 (2) 2 : 3 : 2 (3) 1 : 4 : 1 (4) 3 : 4 : 3

7. ABCD is a rectangle piece of paper with AB = 8 and BC = 6. The paper is folded such that vertex C coincides with vertex A and a firm crease is formed. Find the length of the crease.

   (1) 10 (2) 8 (3) 7.5 (4) 6

8. ABCD is a square inscribed in right angle triangle EBF such that AB and BC lie on the perpendicular legs of the triangle and vertex D lies on the hypotenuse. If EB = 6 and BF = 8, find the side of the square.

   (1) 6/7 (2) 8/7 (3) 32/7 (4) 24/7

9. In right angle triangle ABC, with B being the right angle, point D divides BC in ratio 2 : 1 and point E divides BA in ratio 1 : 2. DF and EG are perpendiculars drawn on the hypotenuse. Find length FG, if AB = 9 and BC = 12.

   (1) 8.2 (2) 7.5 (3) 7.2 (4) 6.4

10. In right angle triangle ABC, with B being the right angle, BD is drawn perpendicular to AC. E and F are the centers of the in-circles drawn in triangles ABD and BDC respectively. If AB = 15, BC = 20, find EF.

   (1) 7 (2) $\sqrt{50}$ (3) 5 (4) $\sqrt{35}$

Quadrilaterals

There are hardly any questions based on quadrilaterals alone. They usually find an appearance along with some theory of triangles or circles, as we have seen in the questions based on triangles. So the following test briefly captures the salient points of the various quadrilaterals…

Parallelogram

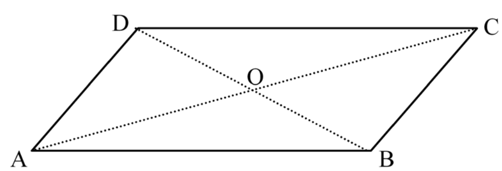
Any one of the following conditions is sufficient in itself to define a parallelogram

Both the pairs of opposite sides are parallel

One pair of opposite sides are parallel and equal

The diagonals of the quadrilateral bisect each other

Thus, if in any quadrilateral the diagonals bisect each other, it necessarily has to be a parallelogram.



Further it would be worthwhile to keep the following in mind about a parallelogram…

Opposite sides are equal in length

Opposite angles are equal and adjacent angles are supplementary.

Area of a parallelogram = base × height

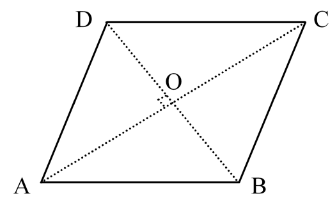
The areas of triangles ABC and ADB are equal as they stand on the same base and have the same height.

Since the diagonals bisect each other, BO will be a median in triangle ABC and similar results exists for other half of diagonals. Thus one can apply Apollonius Theorem in questions involving lengths of the diagonals.

A common misconception is that the diagonals are also the angle bisectors. The diagonals need not be the angle bisectors in a parallelogram.

Rhombus

In a parallelogram, when the lengths of the adjacent sides are equal, it becomes a rhombus. Thus rhombus is a specific case of parallelogram and as such all the properties of a parallelogram would also be valid for a rhombus.



Additionally because of the specific case of adjacent sides being equal, we have the following additional properties..

The diagonals are going to bisect each other at right angles.

The diagonals are going to be the angle bisectors as well.

The area of a rhombus is $\frac{1}{2}d_1d_2$ where d_1 and d_2 are the lengths of the diagonals. (In fact this formula for area is valid for any quadrilaterals where the diagonals intersect at right angles e.g. kite). The earlier formula i.e. base × height, is still valid as rhombus is also a parallelogram.

Rectangle

In a parallelogram when the adjacent sides are perpendicular to each other, it becomes a rectangle. Thus a rectangle is also a specific case of parallelogram and as such all the properties of a parallelogram would also be valid for a rectangle as well.

The only additional property as distinct from a parallelogram is that the diagonals would become equal in length. The diagonals need not be the angle bisector (this will happen only when adjacent sides are equal)

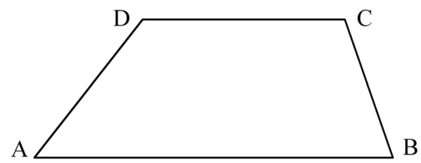
Square

In this case, the adjacent sides are perpendicular to each other and are equal to each other as well. Thus a square is a rhombus, is also a rectangle and obviously parallelogram being the parent figure, a square is also a parallelogram.

Thus, a square will have all the properties of a rhombus as well, specifically, diagonals bisecting at right angles and diagonals being the angle bisector. Similarly, while the area of a square is $(\text{side})^2$, it is also $\frac{1}{2}\times(\text{diagonal})^2$, using the formula for area of a rhombus.

Trapezium

A distinct figure as compared to the above is a trapezium where only one pair of opposite sides is parallel. The other pair of opposite sides, which are not parallel, are called oblique sides.



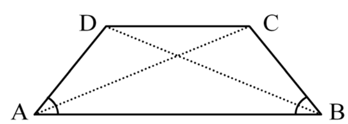
In a trapezium...

1. The opposite angles are not equal. Just the allied interior angles between the parallel lines have to be supplementary.
2. Line joining the mid-points of the oblique sides is parallel to the parallel sides and its length is the arithmetic mean of the lengths of the parallel sides.
3. Area = $\frac{1}{2} \times (\text{sum of parallel sides}) \times \text{height}$.

A specific type of trapezium is an Isosceles Trapezium. In this the lengths of the oblique sides are equal. Because of this, in an isosceles trapezium...

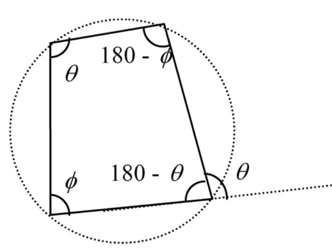
The base angles are equal and so are the other two angles

The diagonals become equal in length



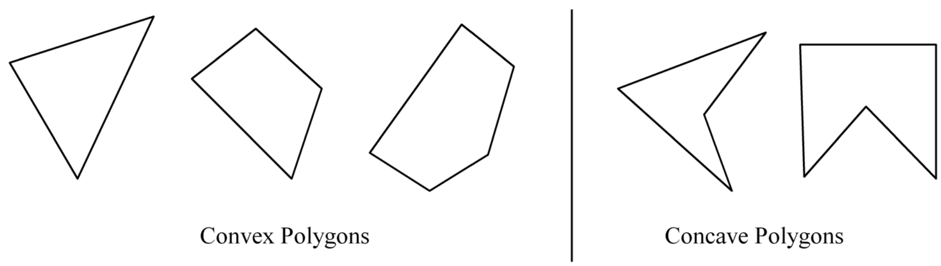
Cyclic Quadrilateral

A cyclic quadrilateral is one in which the four vertices of the quadrilateral lie on a circle. In a cyclic quadrilateral the opposite angles are supplementary (the reason for this will be learnt in the chapter on circles). Whenever it is mentioned that a quadrilateral is cyclic, most often the above property will be used. Another feature worth remembering for a cyclic quadrilateral is that the exterior angle is equal to the sum of remote interior angle.



Polygons

Any closed figure whose sides are straight lines is a polygon. Examples...



If all the vertices lie on the same side of the line containing each of the sides of the polygon, then it is a **Convex Polygon**. If the above is not the case, it will be a **Concave Polygon**.

As far as management entrance exams go, we will focus mainly on convex polygon. So unless specified otherwise, assume the polygon to be convex.

Further polygons are named depending on the number of sides, as follows:

3 sides: Triangle	4 sides: Quadrilateral	5 sides: Pentagon
6 sides: Hexagon	7 sides: Heptagon	8 sides: Octagon
9 sides: Nonagon	10 sides: Decagon	12 sides: Dodecagon

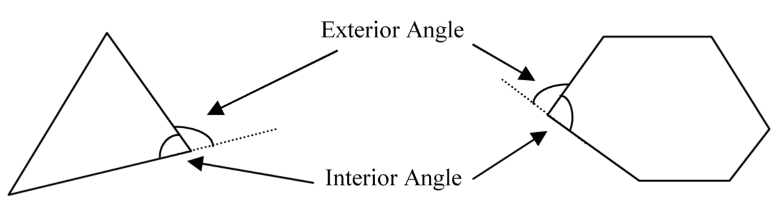
Regular Polygon:

A regular polygon is one in which all the sides are of equal length. Consequently all the interior angles of a regular convex polygon would also be equal.

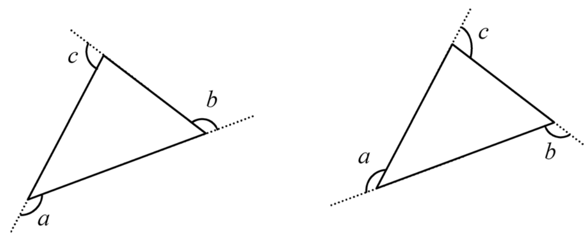
Interior and Exterior Angle:

The following figure depicts a pair of interior and exterior angle for a triangle and a hexagon.

Exterior angle is formed by extending a side of the polygon and the angle formed by the extended side with the adjacent side is called the exterior angle.



A polygon of n sides has n exterior angles. Thus, a triangle will have three exterior angles. The following figure depicts all the three exterior angles. Please note that either of the three triplets can be taken as a set of exterior angles, and it is wrong to state that there are 6 exterior angles.



Sum of all exterior angles of any convex polygon is 360°. This is valid for all convex polygons, irrespective of the number of sides.

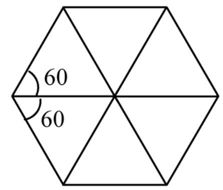
The above can be used to find the sum of all interior angles of any convex polygon. We know a pair of interior and exterior angles would add up to 180°. In a polygon of n sides, there would be n pairs of interior and exterior angle and their sum would be $n \times 180^\circ$. Excluding the sum of all exterior angles i.e. 360, the sum of all interior angles will be $n \times 180^\circ - 360^\circ = (n-2) \times 180^\circ$ or as it is usually expressed $(2n-4) \times 90^\circ$

Thus, one should remember the following facts about Regular polygons...

Polygon	Each exterior angle	Each interior angle	Sum of interior angles
Triangle	$\frac{360}{3} = 120^\circ$	60°	180°
Quadrilateral	$\frac{360}{4} = 90^\circ$	90°	360°
Pentagon	$\frac{360}{5} = 72^\circ$	108°	540°
Hexagon	$\frac{360}{6} = 60^\circ$	120°	720°
Octagon	$\frac{360}{8} = 45^\circ$	135°	1080°

Of these polygons, the next popular one after triangles and quadrilaterals, are hexagons. So please keep the following in mind while dealing with a regular hexagon...

A regular hexagon can be considered as 6 equilateral triangles placed side by side....



Thus, if a is the side of the hexagon, then the area of the hexagon = $6 \times \frac{\sqrt{3}}{4} a^2$

Exercise 8

1. The lengths of the diagonals of a parallelogram are 16 units and 30 units. If the length of one side of the parallelogram is 17 units, what is the perimeter of the parallelogram?

   (1) 64 (2) 66 (3) 68 (4) 72

2. In a rhombus, if the two diagonals measure 24 units and 32 units, find the perimeter of the rhombus.

   (1) 40 (2) 80 (3) 120 (4) 160

3. In rectangle ABCD, points P, Q, R and S divide the sides AB, CB, CD and AD in the ratio 2 : 3. Find the ratio of the area of quadrilateral PQRS to the area of rectangle ABCD.

   (1) 16 : 25 (2) 19 : 25 (3) 4 : 9 (4) 12 : 25

4. In parallelogram ABCD, the midpoints of AB, BC, CD and AD are joined to form another quadrilateral PQRS. If the area of quadrilateral PQRS is a sq units, what is the area of the parallelogram (in terms of a)?

   (1) a (2) $2a$ (3) $4a$ (4) $8a$

5. Square ABCD, with side = 3 cm, is rotated by 45 degree keeping its center fixed to result into another square PQRS. What is the area of the region common to the two squares?

   (1) 7 (2) $9 \times \frac{1+\sqrt{2}}{2+\sqrt{2}}$ (3) $9 \times \frac{1+2\sqrt{2}}{3+2\sqrt{2}}$ (4) $9 \times \frac{2+2\sqrt{2}}{3+2\sqrt{2}}$

6. In parallelogram ABCD, the bisector of angle ABC intersects AD at point P. If l(PD) = 5, l(BP) = 6, and l(CP) = 6, find the length of AB.

   (1) 3 (2) 4 (3) 5 (4) 6

7. In isosceles trapezium ABCD, AB and CD are the parallel sides and have lengths equal to 16 cm and 10 cm. If the length of oblique sides is 5 cm, find the area of the trapezium.

   (1) 65 (2) 48 (3) 52 (4) 56

8. In a trapezium ABCD with AB || CD, EF is a line parallel to the parallel sides with E and F lying on AD and BC such that the perimeter of trapezium ABFE and EFCD are equal. If AB = 18, BC = 6, CD = 13 and AD = 4, find the ratio AE : ED.

   (1) 1 : 2 (2) 1 : 3 (3) 1 : 4 (4) 2 : 3

9. If the ratio of interior angles of two regular polygons is 75 : 78 and the difference in the number of sides of the two polygons is 3, then find the ratio of the number of sides of the polygons.

   (1) 5 : 4 (2) 4 : 3 (3) 4 : 5 (4) 3 : 4

10. If the difference between the sum of all interior angles of two polygons is 720, find the difference between the number of sides of the two polygons.

    (1) 4 (2) 5 (3) 6 (4) 7

11. Each side of a given polygon is parallel to either the X-axis or the Y-axis. A corner of such a polygon is said to be convex if the internal angle is 90 degrees or concave if the internal angle is 270 degrees. If the number of convex corners is 25, find the number of concave corners.

    (1) 21 (2) 25 (3) 29 (4) Cannot be determined

12. In a hexagon of unit sides, three alternate vertices are joined to form an equilateral triangle within the hexagon. Find the area of the triangle so formed?

    (1) $\sqrt{3}$ (2) $\frac{3\sqrt{3}}{2}$ (3) $\frac{3\sqrt{3}}{4}$ (4) $\frac{3}{4}$

Circles:

Circle is the locus of points that are equidistant from a given point. The given point is the center of the circle and the distance of the equidistant points from the center is the radius of the circle.

Circumference, Area, Sector, Segment

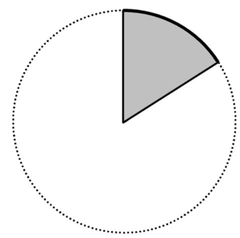
In a circle of radius r,

Circumference = $2\pi r$

Area = πr^2

Segment is a part of the circumference, shown by bold line in the adjoining figure.

Length of the segment = $\frac{\theta}{360} \times 2\pi r$



Sector is the shaded area shown in the figure

Area of a sector = $\frac{\theta}{360} \times \pi r^2$

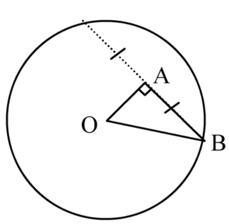
Chord

A chord is a line segment joining any two points on the circumference. The longest chord is the diameter and as the chord moves away from the center, the length of the chord decreases.

The line joining the mid-point of the chord to the center is perpendicular to the chord.

Conversely, **perpendicular dropped from the center to the chord bisects the chord.**

The above property is invariably used in questions related to a chord. Thus one should be conversant with the fact that while dealing with the length of a chord, one would get a right angle triangle with the radius of the circle being the hypotenuse and the perpendicular sides being half the length of the chord and the distance of the chord form the center...

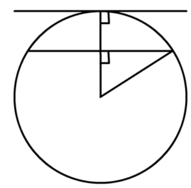


Hypotenuse : OB, radius
Perpendicular sides : OA, Distance of chord from center
AB, Half the chord length

E.g. 26: What is the distance between two parallel chords of lengths 32 cm and 24 cm in a circle of radius 20 cm?

Radius of 20 cm will be the hypotenuse and one of the perpendicular sides will be half the length of the chords i.e. in one case it will be 16 and in other casse it will be 12. Thus, the other perpendicular sides i.e. distance of chord from center will be 12 in first case and 16 in second case (because 12, 16, 20 is a pythagorean triplet). Thus, distance between the chords could be
12 + 16 = 28 (if both the chords are on opposite side of the center) or
16 – 12 = 4 (if both chords are on same side of center)

E.g. 27: APB is a tangent drawn to a circle with center O such that P is the point of tangency. CD is a chord of length 18 cm drawn parallel to AB at a distance of 3 cm from P. Find the radius of the circle.



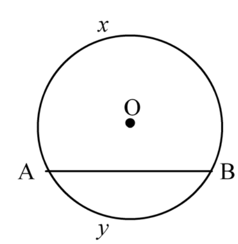
OP is perpendicular to the tangent and since the chord is parallel to the tangent, OP is also perpendicular to the chord. Thus it will bisect the chord.

If r is the radius of the circle, 9, $r - 3$ and r are the sides of the right angle triangle and thus, $9^2 + (r - 3)^2 = r^2$

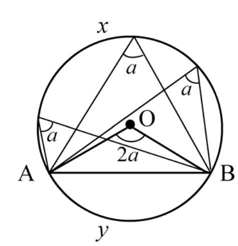
$81 + r^2 - 6r + 9 = r^2$ i.e. $6r = 90$ i.e. $r = 15$.

Angles in a segment and angle at center.

The chord AB divides the circle into two segments viz, the major segment AxB and the minor segment AyB.

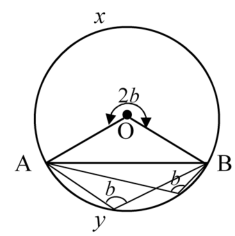


An angle formed in segment AxB, is where the vertex of the angle is on the segment AxB and the rays of the angle pass through A and B. Many such angles can be formed as seen in the figure.

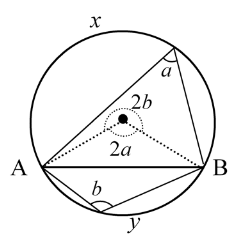


All angles formed in a segment are equal. Further the angle formed by the chord at the center is twice the angle formed by the chord at the circumference

Similarly all angles formed in the minor segment A*y*B will also be equal and in this case also, the angle at the center will be twice the angle at the circumference, but in this case the angle at the center will be the reflex angle (angle in a minor segment is obtuse and hence twice the angle will be more than 180 degrees).

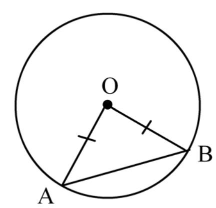


When angles are formed in both the minor and major segment, we get a cyclic quadrilateral and as learnt earlier we can see that the sum of opposite angles is 180 degrees.

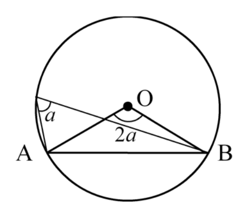


There are two approaches in solving questions based on angles being formed at the intersection of chords...

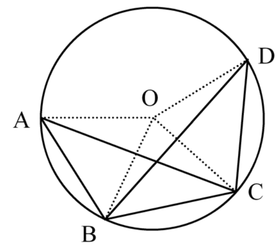
Approach 1: A triangle with one vertex being the center of the circle and other two vertices lying on the circumference will be a isosceles triangle as two of its sides are equal to the radius of the circle. So, in such a triangle knowing any one angle can help us find the values for both the other angles..



Approach 2: Also look out for angles being formed at the circumference because they can be easily found if one knows the angle being formed by the same chord at the center, see diagram for identifying such patterns...



E.g. 28: Chord AB, BC and CD subtend angle $60^\circ, 70^\circ$ and 80° at the center of the circle. Find the obtuse angle at the intersection of AC and BD.



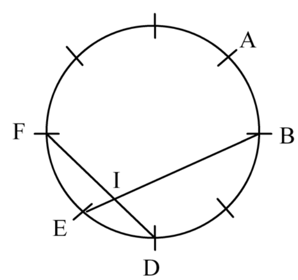
The required angle is formed in the interior of the circle i.e. it is not formed at the center and nor is it formed at the circumference. It is not possible to find this angle directly, we would have to find the angle by finding other angles formed in a triangle which includes the angle to be found.

Chord AB forms 60 degrees at the center and hence it would form 30 degrees at the circumference. Thus $\angle ACB = 30^\circ$.

Similarly, chord CD forms 80 degrees at the center and hence it would form 40 degrees at the circumference. Thus $\angle DBC = 40^\circ$.

Thus the required angle = 180 – 30 – 40 = 110 degrees.

E.g. 29: A circle is divided into eight equal segments. The points are numbered A, B, C, ..., H in a clockwise manner, and then points B and E are joined and D and F are joined. Find the measure of $\angle DIE$, where I is the intersection of the two lines.



As seen in the earlier example, the required angle is in the interior and not at the circumference. So we have to find it by finding the other angles at the circumference.

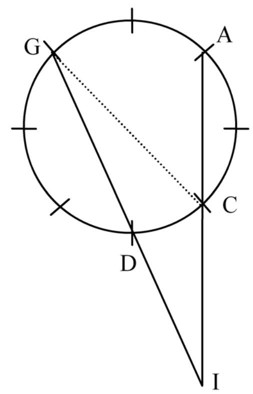
Each of the eight segment of the circle subtends angle 45 degrees at the center. Thus $\angle BED$ is formed by chord BD at the circumference and hence would be half of the angle subtended by BD at the center i.e. half of 90 degrees. Thus, $\angle BED = 45^\circ$.

Similarly, $\angle FDE$ is formed by chord FE at the circumference and would be half of the angle formed by FE at the center i.e. half of 45 degrees. Thus, $\angle FDE = 22.5^\circ$.

So, $\angle DIE = 180 - 45 - 22.5 = 112.5^\circ$.

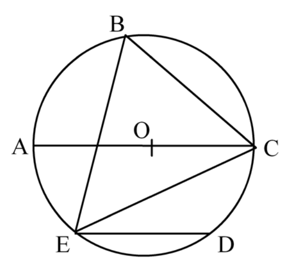
As seen in the earlier two examples, the measure of angle being formed in the interior of a circle (not at the circumference or the center) is found in an indirect way by finding the angles formed at the circumference. The same approach can also be used to find the measure of an angle formed at the exterior of a circle...

E.g. 30: In the above example, find the measure of angle formed at the intersection of AC and GD.



$\angle CGD = 22.5^\circ$ and $\angle GCA = 45^\circ$. Since $\angle GCA$ is an external angle to triangle CGI, it is equal to the sum of $\angle CGD$ and $\angle CID$. Thus $\angle CID = 22.5^\circ$.

E.g. 31: In the adjoining figure chord ED is parallel to the diameter AC of the circle. If $\angle CBE = 65^\circ$, then what is the value of $\angle DEC$?



Since $\angle CBE = 65^\circ$, $\angle COE = 130^\circ$.

In triangle COE, since CO = OE, $\angle OCE = 25^\circ$

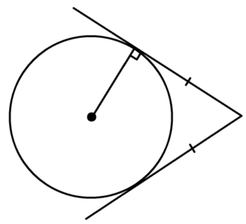
Since ED is parallel to AC, $\angle DEC = 25^\circ$, being the alternate angle.

Tangents

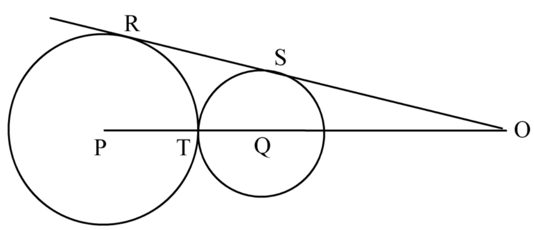
A tangent is line that touches the circle at only one point. The point where the line touches the circle is called point of tangency. Two aspects of a tangent that are going to be used very often are...

1. The line joining the center and the point of tangency is perpendicular to the tangent.

2. From an external point, two tangents can be formed to a circle. The length of the two tangents are equal.



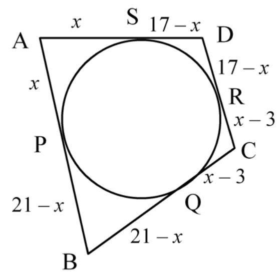
E.g. 32: In the adjoining figure, circles with centers P and Q touch each other at T and have a common tangent that touches them at points R and S respectively. This common tangent meets the line joining P and Q at O. If the diameters of the circles are in the ratio 4 : 3 and PQ = 28, find length OT.



Triangles ORP and OSQ are similar since $\angle O$ is common to both the triangles and $\angle ORP = \angle OSQ = 90^\circ$. Thus the ratio of the corresponding sides is same as the ratio of RP and SQ i.e. the radii i.e. given to be 4 : 3. Thus if OQ = $3k$, then OP = $4k$ and we have $4k - 3k = 28$ i.e. $k = 28$. Thus OQ = $3k$ = 84.

Further PQ is the sum of the radii and since the radii are in ratio 4 : 3, hence PT = $\frac{4}{7} \times 28 = 16$ and QT = $\frac{3}{7} \times 28 = 12$. Thus, OT = 84 + 12 = 96.

E.g. 33: A quadrilateral is circumscribed about a circle. If three sides of the quadrilateral are 17, 18, and 21, not necessarily in that order, compute the smallest possible value of the fourth side.



If
AB = 21,
BC = 18
AD = 17

The shortest measure for the fourth side could happen in any of the situation when it is opposite to side measuring 17 or 18 or 21. Thus, we need to check three conditions. One such condition is as shown in the figure when we are finding the side opposite to 21.

In all cases, tangents from an external point are equal and assuming length of any tangent to be x, we can get length of all other tangents in terms of x and can find the length of the fourth side.

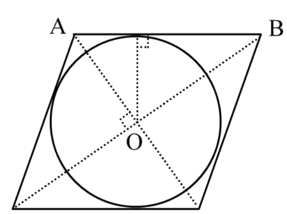
If side measuring 21 is between side measuring 17 and 18, as shown in above figure, length of fourth side is $(17 - x) + (x - 3) = 14$.

If side measuring 18 is between side measuring 21 and 17, length of fourth side can be found as 20 and if side measuring 17 is between side measuring 21 and 18, length of fourth side can be found as 22.

Thus, the smallest length for the fourth side is 14 and it occurs when side measuring 21 is between the sides measuring 17 and 18.

E.g. 34: The circle above is inscribed in rhombus ABCD. Segments AC and BD are diagonals of the rhombus and measure 12 and 24 cm respectively. Find the area of the inscribed circle.

We know that in a rhombus, the diagonals bisect at right angles. Thus, the side of the rhombus can be found as $\sqrt{6^2 + 12^2} = \sqrt{180}$.



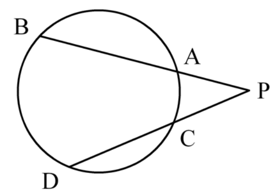
The area of triangle AOB can be found by either considering the side of the rhombus as the base (and the radius being the height) or by considering AO as the base and BO as the height. Equating the area so found,

$\frac{1}{2} \times 12 \times 6 = \frac{1}{2} \times \sqrt{180} \times r \Rightarrow r = \frac{12}{\sqrt{5}}$. Thus, area of the inscribed circle = $\frac{144}{5}\pi$.

Secants

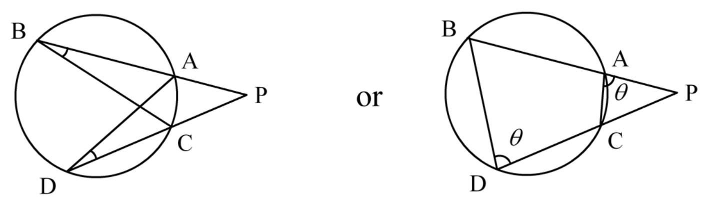
Secant is the line of which a segment is a chord.

The following property will be very handy...

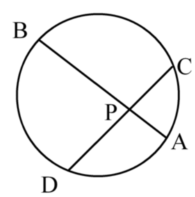


PA × PB = PC × PD

While the proof of the above is not necessary, it is worth noting it because of the use of similarity in the proof. So pay attention to the pattern of similar triangles formed in a circle...

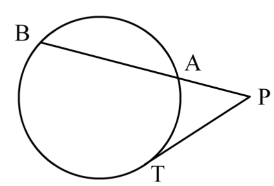


The above is property is valid also for an internal intersection of chords...



PA × PB = PC × PD

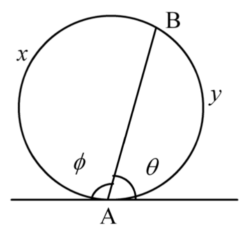
A similar property can also be constructed in the case of a secant and a tangent...



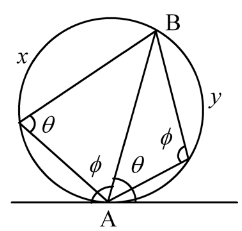
PT^2 = PA × PB

Tangent Secant Theorem
(also called Alternate Segment Theorem)

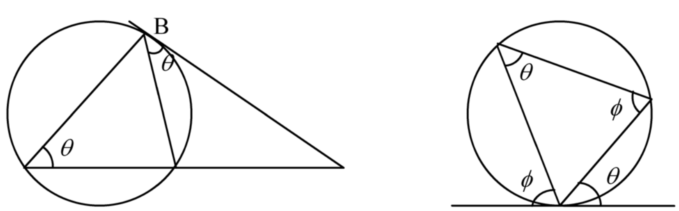
This theorem relates the angle formed between a tangent and a secant drawn at the point of tangency with the angle formed in the alternate segment. Thus, in the following figure, the angle θ is going to be related to angle formed in the 'alternate' segment i.e. in segment A*x*B and the angle φ is related to angle formed in the alternate segment i.e. in segment A*y*B.



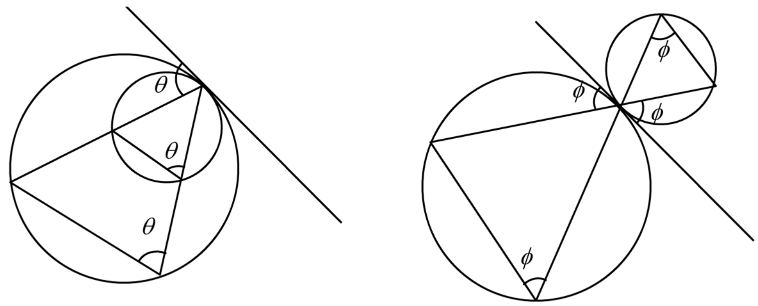
Angle between the tangent and a secant drawn at point of tangent is equal to the angle formed in the alternate segment. Thus,



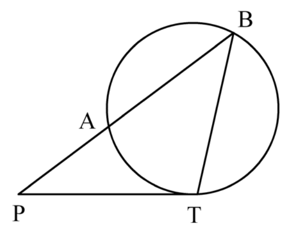
The two most common figures in which one has to use tangent secant theorem are...



Another situation where the tangent secant theorem comes in very handy is in the case of two circles that touch each other either internally or externally. The following figures explain how tangent secant theorem can be applied in these situations after drawing a common tangent to the two circles at the point of contact.



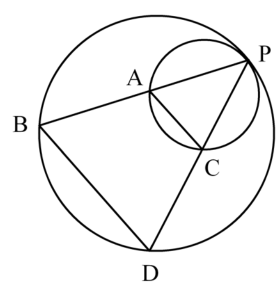
E.g. 35: PT is a tangent to a circle at T as shown in the adjoining figure (not drawn to scale). PAB is a secant intersecting the circle at A and B as shown. If PT = BT = 6 and PB = 9, find length of AT.



In triangles PTA and PBT, $\angle \text{PTA} = \angle \text{PBT}$ (by tangent secant theorem) and $\angle \text{P}$ is common to the two triangles and hence the triangles are similar.

Thus, $\frac{\text{AT}}{\text{BT}} = \frac{\text{PT}}{\text{PB}} \Rightarrow \text{AT} = 4$.

E.g. 36: In the figure shown (not drawn to scale) the two circles touch each other at P. If AC : BD is 2 : 5, find the ratio of $\frac{l(\text{PA}) \times l(\text{PB})}{l(\text{PC}) \times l(\text{PD})}$.

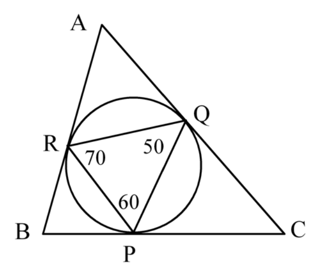


$\angle \text{ACP}$ and $\angle \text{BDP}$ will be equal as each will be equal to the angle between the secant BP and the common tangent at P (not drawn). Further $\angle \text{P}$ is common to the two triangles

Thus the two triangle PAC and PBD are similar and the ratio of the corresponding sides AC and BD is given to be 2 : 5. Thus, AP : BP and CP : DP will also be 2 : 5. Thus if AP = $2k$, then BP = $5k$ and if CP = $2n$, then DP = $5n$.

So the required ratio will be $\frac{2k \times 5k}{2n \times 5n} = \frac{k^2}{n^2}$ i.e. the answer cannot be determined with just this data.

E.g. 37: The in-circle of triangle ABC touches the three sides of the triangle at P, Q and R. Line PQ, QR and RP are joined to form a triangle PQR, for which the given circle is the circum-circle. Find the measures of the internal angles of triangle ABC if the measures of the internal angles of triangle PQR are 50, 60 and 70 degrees.



Using tangent-secant theorem,

$\angle AQR = \angle ARQ = 60^\circ \Rightarrow \angle A = 60^\circ$

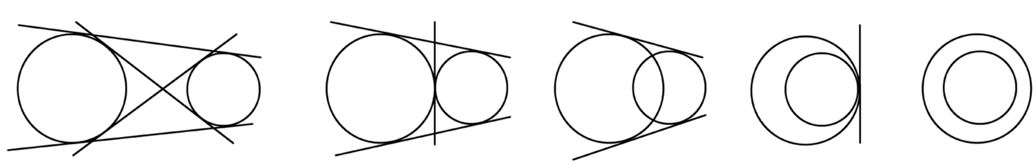
$\angle BPR = \angle BRP = 50^\circ \Rightarrow \angle B = 80^\circ$

$\angle CQP = \angle CPQ = 70^\circ \Rightarrow \angle C = 40^\circ$

Thus angles of triangle ABC are 40, 60 and 80 degrees.

Common Tangents to Two Circles

The number of common tangents to two circles could range from 4 to 0 depending on the relative placement of the circles...



Case (i): Circles external to each other. 2 Direct and 2 Transverse Tangent

Case (ii): Circles touch externally. 2 Direct and 1 Transverse Tangent

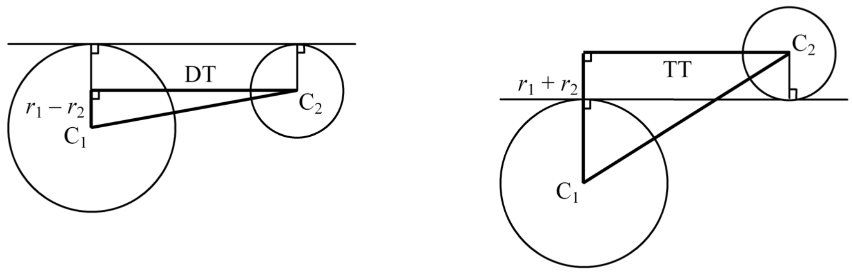
Case (iii): Circles intersect. 2 Direct Tangent

Case (iv): Circles touch internally. 1 Transverse Tangent

Case (v): Circle within other. No common tangent

A common tangent is said to be a direct common tangent if the two circles lie on the same side of the tangent and is a transverse tangent if the circles lie on the opposite sides of the tangent.

An expression to find the length of direct or transverse common tangent can be found as follows...

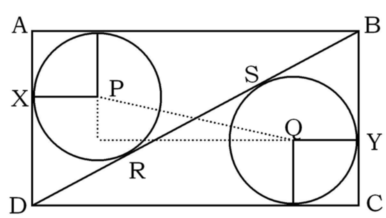


As seen in the above figures, the length of the Direct Common Tangent, DT, is given by $DT = \sqrt{C_1C_2^{\,2} - (r_1 - r_2)^2}$ and the length of Transverse Common Tangent, TT, is given by $TT = \sqrt{C_1C_2^{\,2} - (r_1 + r_2)^2}$, where C_1C_2 is the distance between the centers and r_1 and r_2 are the radius of the two circles.

One should not try to memorise the above expressions and should simply think of a right angle triangle with hypotenuse being the distance between the centers and the perpendicular sides being length of the tangent and $r_1 \pm r_2$.

A difficult example based on the above is...

E.g. 38: In rectangle ABCD, AB = 8 cm and BC = 6 cm. P and Q are the centers of the in-circles of triangle ABC and ADC. The circles touch the diagonal AC at R and S. Find the length RS.



RS is the length of the transverse tangent and hence to find its length one needs to know the distance between the centers and the radius of the two circles. Let's find the distance between the centers, PQ, first.

Each of the triangles is a right angle triangle with sides 6, 8, and 10. As learnt earlier the in-radius of such a triangle is 2 cm. Thus, the horizontal distance between the centers is 8 – 2 – 2 = 4 and the vertical distance between the centers is 6 – 2 – 2 = 2. Thus, $PQ = \sqrt{4^2 + 2^2} = \sqrt{20}$.

Now, we can find the length of the transverse tangent, RS, as

$$PQ = \sqrt{\left(\sqrt{20}\right)^2 - (2+2)^2} = \sqrt{20-16} = 2.$$

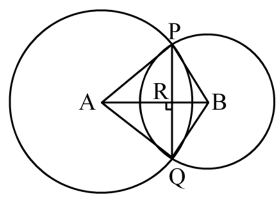
Alternate Elegant Approach:

DX = DR = BY = BS = 4 (Tangents from external point are equal and vertical distance DX or BY = 6 – 2 = 4)

Thus, RS = DB – DR – BS = 10 – 4 – 4 = 2

Common Chord to Two Intersecting Circles

In the following figure where the two circles with centers A and B intersect each other, the chord PQ is the common chord.



A few aspects of the above figure worth keeping in mind are...

1. The line joining the centers (AB) bisects the common chord at right angles. Thus R is the mid-point of PQ and also $AB \perp PQ$.

2. R need not necessarily be the midpoint of AB. In fact R will divide AB in the ratio of $r_1 : r_2$.

3. $\angle PAB = \angle QAB$; $\angle PBA = \angle QBA$; $\angle APR = \angle AQR$; $\angle BPR = \angle BQR$; $\angle APB = \angle AQB$

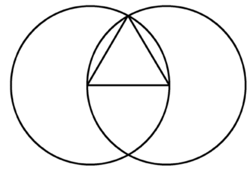
4. AP and AQ need not necessarily be tangents to the circle with center B. Similarly BP and BQ also need not necessarily be tangents to the circle with center A. Thus the angles $\angle APB$ and $\angle AQB$ need not be right angles. Infact when $\angle APB = \angle AQB = 90^\circ$, the circles are said to be intersecting orthogonally.

E.g. 39: Find the length of the common chord of two circles of radii 15 cm and 20 cm, whose centers are 25 cm apart.

Since the lengths given are 15, 20 and 25, it should be obvious that the circles intersect orthogonally and the angle formed by the radii at the intersection of the circles is a right angle. And from this right angle a perpendicular is dropped to the hypotenuse (line joining the centers). Also since the values used 15, 20 and 25 are so very often used in the earlier questions one should have got the length of the perpendicular dropped from the right angle to the hypotenuse as 12 without the need of any calculation.

Since the common chord if bisected by the line joining the centers, the length of the common chord is 24 cm.

E.g. 40: Circle A and Circle B both have a radius of 1 unit. The centers of each circle are 1 unit apart as well. Find the area of the union of the two circles.



It should be obvious that the two circles pass through each other's centers as the radii are 1 unit and the distance between the centers is also 1 unit.

The triangle shown in the figure is a equilateral triangle of area, T = $\frac{\sqrt{3}}{4}$ sq. units. The angle in an equilateral triangle is 60 degrees and hence the area of the sector, S = $\frac{60}{360} \times \pi = \frac{\pi}{6}$.

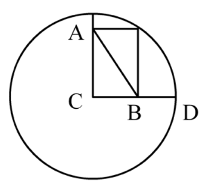
Area of the required union = 2 × T + 4 × (S – T) = 4 × S – 2T = $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.

Exercise 9

1. In a circle, the distance between two parallel chords of length 48 cm and 14 cm (drawn on the same side of the center) is 17. Find the radius of the circle.

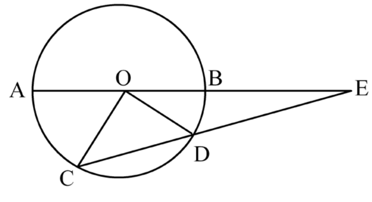
   (1) 50 (2) 25 (3) 30 (4) 40

2. C is the center of the circle below. The length of segment CB is 7 units. The length of segment BD is 4 units. Find the length of diagonal AB in the rectangle.



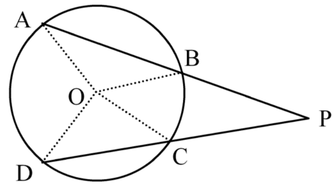
   (1) 11 (2) 12 (3) 15 (4) 16

3. Diameter AOB of circle with center O is extended such that it intersects extended chord CD at E. If $\angle AOC = 50^\circ$ and $\angle AEC = 15^\circ$, find $\angle BOD$.



   (1) 20 (2) 24 (3) 25 (4) 30

4. In the figure below, find the measure of $\angle APD$ if $\angle AOD = 100^\circ$ and $\angle BOC = 50^\circ$.

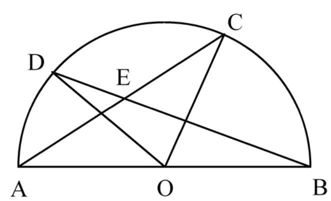


(1) 130 (2) 30 (3) 25 (4) 75

5. The diagonals EC and GB of a regular 10 sided polygon ABCDEFGHIJ are extended to intersect at P. Find the measure of angle P.

(1) 17.5 (2) 18 (3) 22.5 (4) 24

6. AOB is the diameter of the semicircle with center O as shown in the figure. If $m\angle DOC = 80^\circ$, find $m\angle DEC$.

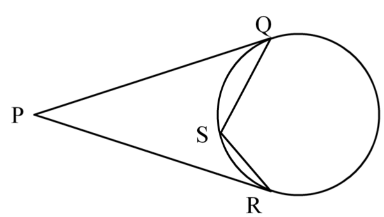


(1) 130 (2) 140 (3) 150 (4) 160

7. The in-circle of triangle ABC touches AB, BC and AC at P, Q and R respectively. If AB = 12 cms, BC = 18 cm and AC = 24. Find AP + BQ + CR.

(1) 26 (2) 27 (3) 28 (4) 30

8. In the following figure PQ and PR are tangents to the circle. If $\angle RPQ = 40^\circ$, find the sum of measure of angles $\angle PQS + \angle PRS$.

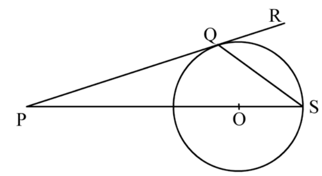


(1) 60 (2) 75 (3) 80 (4) 70

9. Two circles touch each other externally at P. APB and CPD are two straight lines intersecting the first circle at A and C and the second circle at B and D and passing through P. If AP = 12, PC = 4, PD = 6, find length PB.

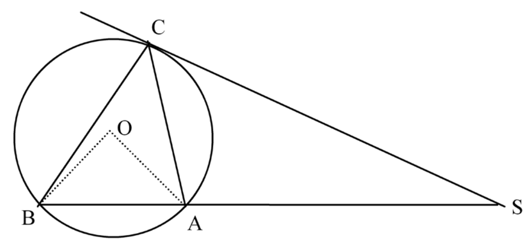
(1) 8 (2) 10 (3) 12 (4) 18

10. In the following figure PQR is a tangent to the circle with center O. If $m\angle RQS = 75^\circ$, find $m\angle QPS$.



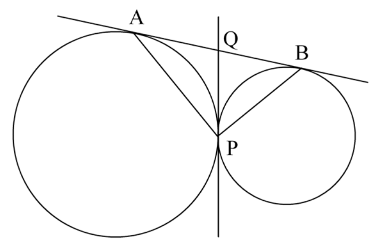
(1) 15 (2) 30
(3) 45 (4) 60

11. In the figure given below (not drawn to scale), A, B and C are three points on a circle with center O. The chord BA is extended to a point S such that CS becomes a tangent to the circle at point C. If angle ASC = 30 and angle ACS = 50 then find the angle $\angle BOA$.



(1) 75 (2) 80
(3) 100 (4) 120

12. In the adjoining figure, if AP = 8 and BP = 6, find the ratio of the radii of the two circles?

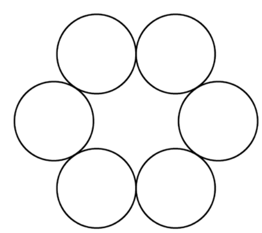


(1) 16 : 9 (2) 4 : 3
(3) 25 : 9 (4) 25 : 16

13. Three coke cans of radius 2 cm are placed such that each can touches the other two cans. The three cans are tied with a string such that the string makes exactly one full round across the three cans and the string is drawn tight. Find the length of the string.

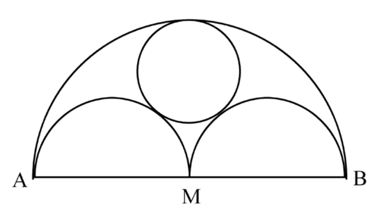
(1) $12(1+\pi)$ (2) $4(1+\sqrt{2})\pi$ (3) $12+4\pi$ (4) $8(1+\sqrt{2})\pi$

14. Six equal circles are placed uniformly such that each touch two other circles and that an equal circle can be placed between the circles, as shown in the figure. Find the area enclosed within the circles if the radius of each circle is 1 unit.



(1) $4\sqrt{3}-2\pi$ (2) $6\sqrt{3}-2\pi$
(3) $6\sqrt{3}-\pi$ (4) $4\sqrt{3}-\pi$

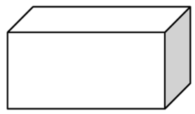
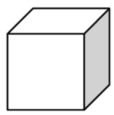
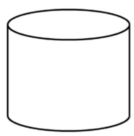
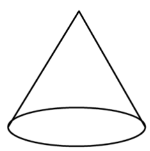
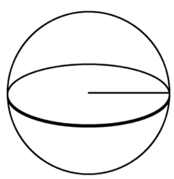
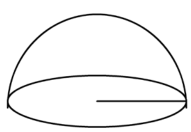
15. In the following figure, AB is the diameter of semicircle with center M. Two semi-circles are drawn with AM and MB as the diameters. A circle is drawn such that it is tangent to all the three semi-circles. If AM = *r* units, find the radius of the circle (in terms of *r*).



(1) $\frac{r}{2}$ (2) $\frac{r}{3}$ (3) $\frac{r}{4}$ (4) $\frac{r}{\sqrt{2}}$

Solids

There are hardly any questions based on solids. The following table gives the formulae for the surface area and volume of the regular solids encountered.

S. No	Name	Figure	Lateral/Curved Surface Area	Total Surface Area	Volume
1.	Cuboid		$2 \times h \times (l + b)$ h – height l – length b – breadth	$2(lb + bh + lh)$	$l \times b \times h$
2.	Cube		$4a^2$ a – edge	$6a^2$	a^3
3.	Right Circular Cylinder		$2\pi rh$ r – radius h – height	$2\pi(r+h)$	$\pi r^2 h$
4.	Right Circular Cone		πrl h – height r – radius l – slant height $l^2 = r^2 + h^2$	$\pi r(r+l)$	$\frac{1}{3} \times \pi r^2 h$
5.	Sphere		$4\pi r^2$ r – radius	$4\pi r^2$ r – radius	$\frac{4}{3} \times \pi r^3$
6.	Hemi-sphere		$2\pi r^2$	$3\pi r^2$	$\frac{2}{3} \times \pi r^3$

Exercise 10

1. A spherical metal ball is melted and is re-casted into smaller spherical metal pellets. If the 64 such smaller pellets could be formed, find the ratio of the surface area of the original metal ball and the total surface area of all the pellets formed.

   (1) 16 : 1 (2) 4 : 1 (3) 1 : 1 (4) 1 : 4

2. A metal in the form of a cylinder of radius 2 cm and height 10 cm is drawn out into a wire with circular cross-section of radius 1 mm. Find the length of the wire so formed.

   (1) 4 mts (2) 40 mts (3) 4000 mm (4) 40 cm

3. The radius of a cylinder is increased by 10%. By what approximate percent should the height of the cylinder be reduced to maintain the volume of the cylinder?

   (1) 10% (2) 12.5% (3) 16.66% (4) 21%

4. A ratio of the sides of a cuboid is 1 : 2 : 3. Further the ratio of the numerical value of its surface area to the numeric value of its volume is 11 : 18. Find the length of the longest diagonal of the cuboid.

   (1) $3\sqrt{14}$ (2) $4\sqrt{14}$ (3) $5\sqrt{14}$ (4) $6\sqrt{14}$

Directions for questions 5 and 6: A spider starts from a point on the bottom edge of a circular cylinder and moves in a spiral manner along the curved surface area. Radius of the cylinder is $\frac{8}{\pi}$ and its height is 15 units.

5. Find shortest distance travelled by the spider if it reaches the top of the cylinder exactly as it completes one circle?

   (1) $\sqrt{465}$ (2) $\sqrt{475}$ (3) $\sqrt{481}$ (4) $\sqrt{491}$

6. What would have been the distance covered by the spider if it would have made exactly $2\frac{1}{2}$ rounds in the time it reaches the top edge.

   (1) $3\sqrt{73}$ (2) $5\sqrt{73}$ (3) $7\sqrt{73}$ (4) $9\sqrt{73}$

7. A cone is formed from a sector of a circle of radius 9 cm and central angle being 120 degrees. Find the volume of the cone so formed.

   (1) $9\sqrt{2}\pi$ (2) $9\sqrt{3}\pi$ (3) $18\sqrt{3}\pi$ (4) $18\sqrt{2}\pi$

8. What is the volume of a tetrahedron of unit side? A tetrahedron is a triangular pyramid with all the triangles being equilateral triangle. Volume of a pyramid = $\frac{1}{3} \times \text{area of base} \times \text{height}$.

   (1) $\frac{1}{3\sqrt{6}}$ (2) $\frac{1}{4\sqrt{6}}$ (3) $\frac{1}{6\sqrt{2}}$ (4) $\frac{1}{3\sqrt{2}}$

9. A rectangular ground has dimensions 16 mts by 30 mts. There is a flagpost at the center of the ground such that the side measuring 30 mts subtends a right angle at the top of the flagpost. Find the height of the flagpost.

   (1) $\sqrt{161}$ (2) $\sqrt{163}$ (3) $\sqrt{165}$ (4) $\sqrt{167}$

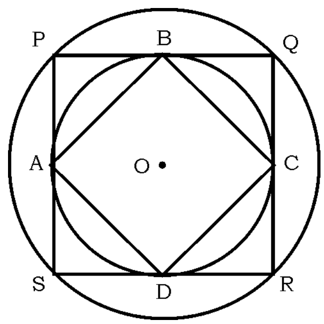
10. A cylinder is carved out of a cone with height 15 cm and radius of base circle 12 cm. What is the maximum volume of the cylinder?

    (1) 80π (2) 160π (3) 320π (4) 640π

CAT Questions

CAT 1999:

1. The figure below shows two concentric circles with centre O. PQRS is a square, inscribed in the outer circle. It also circumscribes the inner circle, touching it at points B, C, D and A. What is the ratio of the perimeter of the outer circle to that of polygon ABCD?



(1) $\frac{\pi}{4}$ (2) $\frac{3\pi}{2}$ (3) $\frac{\pi}{2}$ (4) π

2. There is a circle of radius 1 cm. Each member of a sequence of regular polygons $S_1(n)$, n = 4, 5, 6,....., where n is the number of sides of the polygon, is circumscribing the circle; and each member of the sequence of regular polygons $S_2(n)$, n = 4, 5, 6, ... where n is the number of sides of the polygon, is inscribed in the circle. Let $L_1(n)$ and $L_2(n)$ denote the perimeters of the corresponding polygons of $S_1(n)$ and $S_2(n)$, then $\frac{\{L_1(13)+2\pi\}}{L_2(17)}$ is

(1) greater than $\frac{\pi}{4}$ and less than 1 (2) greater than 1 and less than 2

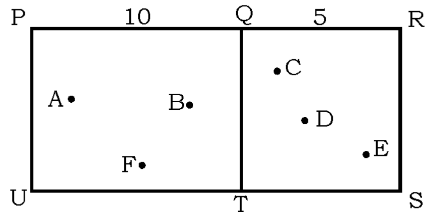
(3) greater than 2 (4) less than $\frac{\pi}{4}$

3. There is a square field with each side 500 metres long. It has a compound wall along its perimeter. At one of its corners, a triangular area of the field is to be cordoned off by erecting a straight line fence. The compound wall and the fence will form its border. If the length of the fence is 100 metres, what is the maximum area in square metres that can be cordoned off?

(1) 2,500 (2) 10,000 (3) 5,000 (4) 20,000

DIRECTIONS : These questions are based on the situation given below.

A rectangle PRSU, is divided into two smaller rectangles PQTU and QRST by the line TQ. PQ = 10 cm, QR = 5 cm and RS = 10 cm. Points A, B, F are within rectangle PQTU and points C, D, E are within the rectangle QRST. The closest pair of points among the pairs (A, C), (A, D), (A, E), (F, C), (F, D), (F, E), (B, C), (B, D), (B, E) are $10\sqrt{3}$ cm apart.



4. Which of the following statements is necessarily true?

   (1) The closest pair of points among the six given points cannot be (F, C)

   (2) Distance between A and B is greater than that between F and C

   (3) The closest pair of points among the six given points is (C, D), (D, E) or (C, E)

   (4) None of the above

5. AB > AF > BF; CD > DE > CE and BF = $6\sqrt{5}$ cm. Which is the closest pair of points among all the six given points?

   (1) B, F (2) C, D (3) A, B (4) None of the above

CAT 2000:

6. What is the number of distinct triangles with integral valued sides and perimeter 14?

   (1) 6 (2) 5 (3) 4 (4) 3

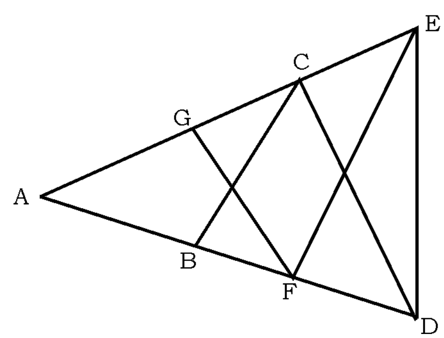
7. Consider a circle with unit radius. There are 7 adjacent sectors, S1, S2, S3,...., S7 in the circle such that their total area is (1/8)th of the area of the circle. Further, the area of the j^{th} sector is twice that of the $(j-1)^{th}$ sector, for j = 2, ..., 7. What the angle, in radians, subtended by the arc of S1 at the centre of the circle?

   (1) $\pi/508$ (2) $\pi/2040$ (3) $\pi/1016$ (4) $\pi/1524$

8. If a, b, c are the sides of a triangle, and $a^2 + b^2 + c^2 = bc + ca + ab$, then the triangle is

   (1) equilateral (2) isosceles (3) right angled (4) obtuse angled

9. In the figure below, AB = BC = CD = DE = EF = FG = GA. Then angle DAE is approximately



   (1) 15° (2) 20°

   (3) 30° (4) 25°

CAT 2001:

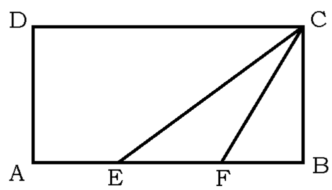
10. A square, whose side is 2 meters, has its corners cut away so as to form an octagon with all sides equal. Then the length of each side of the octagon, in meters is

(1) $\frac{\sqrt{2}}{\sqrt{2}+1}$ (2) $\frac{2}{\sqrt{2}+1}$ (3) $\frac{2}{\sqrt{2}-1}$ (4) $\frac{\sqrt{2}}{\sqrt{2}-1}$

11. A certain city has a circular wall around it, and this wall has four gates pointing north, south, east and west. A house stands outside the city, three kms north of the north gate, and it can just be seen from a point nine kms east of the south gate. What is the diameter of the wall that surrounds the city?

(1) 6 km (2) 9 km (3) 12 km (4) None of these

12. In the below diagram, ABCD is a rectangle with AE = EF = FB. What is the ratio of the area of the triangle CEF and that of the rectangle?



(1) $\frac{1}{6}$ (2) $\frac{1}{8}$ (3) $\frac{1}{9}$ (4) None of these

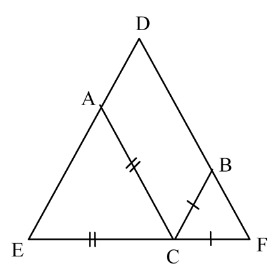
13. A ladder leans against a vertical wall. The top of the ladder is 8 m above the ground. When the bottom of the ladder is moved 2 m farther away from the wall, the top of the ladder rests against the foot of the wall. What is the length of the ladder?

(1) 10 m (2) 15 m (3) 20 m (4) 17 m

14. Euclid has a triangle in mind. Its longest side has length 20 and another of its sides has length 10. Its area is 80. What is the exact length of its third side?

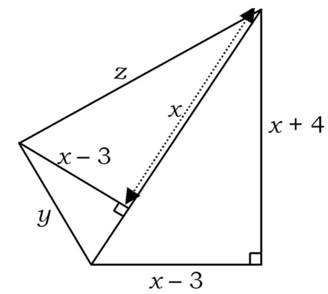
(1) $\sqrt{260}$ (2) $\sqrt{250}$ (3) $\sqrt{240}$ (4) $\sqrt{270}$

15. In triangle DEF shown below, points A, B, and C are taken on DE, DF and EF respectively such that EC = AC and CF = BC. If angle D = 40 degrees then what is angle ACB in degrees?



(1) 140 (2) 70 (3) 100 (4) None of these

16. Based on the figure below, what is the value of x, if $y = 10$.



(1) 10 (2) 11

(3) 12 (4) None of these

17. A rectangular pool 20 metres wide and 60 metres long is surrounded by a walkway of uniform width. If the total area of the walkway is 516 square metres, how wide, in metres, is the walkway?

(1) 43 (2) 4.3 (3) 3 (4) 3.5

CAT 2002:

18. Instead of walking along two adjacent sides of a rectangular field, a boy took a short cut along the diagonal and saved a distance equal to half the longer side. Then the ratio of the shorter side to the longer side is:

(1) 1/2 (2) 2/3 (3) 1/4 (4) 3/4

19. The area of the triangle whose vertices are (a, a), (a + 1, a + 1), (a + 2, a) is:

(1) a^3 (2) 1 (3) 2a (4) $2^{1/2}$

20. Four horses are tethered at four corners of a square plot of side 14 metres (m) so that the adjacent horses can reach one another. There is a small circular pond of area 20 m^2 at the centre. The area left ungrazed is:

(1) 22 m^2 (2) 42 m^2 (3) 84 m^2 (4) 168 m^2

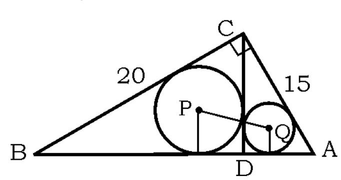
21. The length of the common chord of two circles of radii 15 cm and 20 cm, whose centres are 25 cm apart, is (cm):

(1) 24 (2) 25 (3) 15 (4) 20

22. In a triangle ABC, the internal bisector of the angle A meets BC at D. If AB = 4, AC = 3 and A = 60 degrees, then length of AD is:

(1) $2\sqrt{3}$ (2) $\frac{12}{7}\sqrt{3}$ (3) $\frac{15}{8}\sqrt{3}$ (4) $\frac{6}{7}\sqrt{3}$

23. In the below figure, ACB is a right angled triangle. CD is the altitude. Circles are inscribed within the triangle ACD, BCD. P and Q are the centres of the circles. The distance PQ is

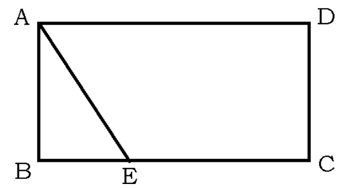


(1) 5 (2) $\sqrt{50}$ (3) 7 (4) 8

24. Neeraj has agreed to mow the front lawn, which is a 20 m by 40 m rectangle. The mower mows a 1 m wide strip. If Neeraj starts at one corner and mows around the lawn toward the center, about how many times would he go round before he has mowed half the lawn?

(1) 2.5 (2) 3.5 (3) 3.8 (4) 4.0

25. In the figure given below, ABCD is a rectangle. The area of the isosceles right triangle ABE = 7 cm^2. (EC) = 3 × (BE). The area of ABCD (in cm^2) is:



(1) 21 (2) 28 (3) 42 (4) 56

CAT 2003 Leaked:

26. Let A and B be two solid spheres such that the surface area of B is 300% higher than the surface area of A. The volume of A is found to be k% lower than the volume of B. The value of k must be___.

(1) 85.5 (2) 92.5 (3) 90.5 (4) 87.5

Directions for questions 27 to 29: A city has two perfectly circular and concentric ring roads, the outer ring road (OR) being twice as long as the inner ring road (IR). There are also four (straight line) chord roads from E_1, the east end point of OR to N_2, the north end point of IR; from N_1, the north end point of OR to W_2, the west end point of IR; from W_1, the west end point of OR, to S_2, the south end point of IR; and from S_1, the south end point of OR to E_2, the east endpoint of IR. Traffic moves at a constant speed of 30π km/hr on the OR road, 20π km/hr on the IR road, and $15\sqrt{5}$ km/ hr on all the chord roads.

27. The ratio of the sum of the lengths of all chord roads, to the length of the outer ring road is___.

(1) $\sqrt{5}:2$ (2) $\sqrt{5}:\sqrt{2}$ (3) $\sqrt{5}:\pi$ (4) None of these.

28. Amit wants to reach N_2 from S_1. It would take him 90 minutes if he goes on minor arc $S_1 – E_1$ on OR, and then on the chord road $E_1 – N_2$. What is the radius of the outer ring road in km?

(1) 60 (2) 40 (3) 30 (4) 20

29. Amit wants to reach E_2 from N_1 using first the chord $N_1 – W_2$ and then the inner ring road. What will be his travel time in minutes on the basis of information given in the above question?

(1) 60 (2) 45 (3) 90 (4) 105

Directions for questions 30 & 31:

Choose your answer as (1) if the question can be answered by one of the statements A or B alone but not by the other.

Choose your answer as (2) if the question can be answered by using either statements A or B alone.

Choose your answer as (3) if the question can be answered by using both the statements A and B together, but cannot be answered by using either statement alone.

Choose your answer as (4) if the question cannot be answered even by using both the statements together.

30. AB is a chord of a circle. AB = 5 cm. A tangent parallel to AB touches the minor arc AB at E. What is the radius of the circle?

    Statement A: AB is not a diameter of the circle.

    Statement B: The distance between AB and the tangent at E is 5 cm.

31. D, E, F are the mid-points of the sides AB, BC and CA of triangle ABC respectively. What is the area of DEF in square centimetres?

    Statement A: AD = 1 cm, DF =1 cm and perimeter of DEF = 3 cm

    Statement B: Perimeter of ABC = 6 cm, AB = 2 cm, and AC = 2 cm.

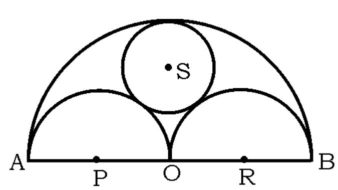
32. Each side of a given polygon is parallel to either the X or the Y axis. A corner of such a polygon is said to be convex if the internal angle is 90° or concave if the internal angle is 270°. If the number of convex corners in such a polygon is 25, the number of concave comers must be___.

    (1) 20 (2) 0 (3) 21 (4) 22

33. There are two concentric circles such that the area of the outer circle is four times the area of the inner circle. Let A, B and C be three distinct points on the perimeter of the outer circle such that AB and AC are tangents to the inner circle. If the area of the outer circle is 12 square centimetres then the area (in square centimetres) of the triangle ABC would be___.

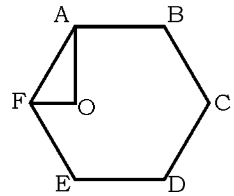
    (1) $\pi\sqrt{12}$ (2) $\frac{9}{\pi}$ (3) $\frac{9\sqrt{3}}{\pi}$ (4) $\frac{6\sqrt{3}}{\pi}$

34. Three horses are grazing within a semi-circular field. In the diagram given below, AB is the diameter of the semi-circular field with centre at O. Horses are tied up at P, R and S such that PO and RO are the radii of semi-circles with centres at P and R respectively, and S is the centre of the circle touching the two semicircles with diameters AO and OB. The horses tied at P and R can graze within the respective semi-circles and the horse tied at S can graze within the circle centred at S. The percentage of the area of the semi-circle with diameter AB that cannot be grazed by the horses is nearest to



    (1) 20 (2) 28 (3) 36 (4) 40

35. In the figure below, ABCDEF is a regular hexagon and angle AOF = 90°. FO is parallel to ED. What is the ratio of the area of the triangle AOF to that of the hexagon ABCDEF?



(1) $\frac{1}{12}$ (2) $\frac{1}{6}$ (3) $\frac{1}{24}$ (4) $\frac{1}{18}$

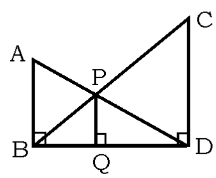
36. A vertical tower OP stands at the centre O of a square ABCD. Let h and b denote the lengths OP and AB respectively. Suppose angle APB = 60°. Then the relationship between h and b can be expressed as___.

(1) $2b^2 = h^2$ (2) $2h^2 = b^2$ (3) $3b^2 = 2h^2$ (4) $3h^2 = 2b^2$

37. In a triangle ABC, AB = 6, BC = 8 and AC = 10. A perpendicular dropped from B, meets the side AC at D. A circle of radius BD (with centre B) is drawn. If the circle cuts AB and BC at P and Q respectively, then AP : QC is equal to___.

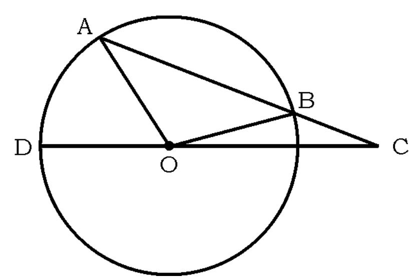
(1) 1 : 1 (2) 3 : 2 (3) 4 : 1 (4) 3 : 8

38. In the diagram given below, angle ABD = angle CDB = angle PQD = 90°. If AB : CD = 3 : 1, the ratio of CD : PQ is:



(1) 1 : 0.69 (2) 1 : 0.75 (3) 1 : 0.72 (4) None of these.

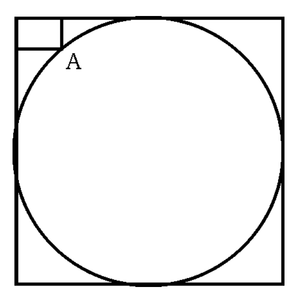
39. In the figure given below, AB is the chord of a circle with centre O. AB is extended to C such that BC = OB. The straight line CO is produced to meet the circle at D. If angle ACD = y ° and angle AOD = x° such that $x = ky$, then the value of k is___.



(1) 3 (2) 2

(3) 1 (4) None of these

40. In the figure below, the rectangle at the corner measures 10 cm × 20 cm. The corner A of the rectangle is also a point on the circumference of the circle. What is the radius of the circle in cm?



(1) 10 cm (2) 40 cm

(3) 50 cm (4) None of these.

CAT 2003 Retest:

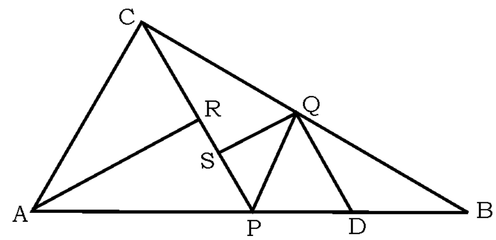
41. Let ABCDEF be a regular hexagon. What is the ratio of the area of the triangle ACE to that of the hexagon ABCDEF?

    (1) $\frac{1}{3}$ (2) $\frac{1}{2}$ (3) $\frac{2}{3}$ (4) $\frac{5}{6}$

42. The length of the circumference of a circle equals the perimeter of a triangle of equal sides, and also the perimeter of a square. The areas covered by the circle, triangle, and square are *c*, *t*, and *s*, respectively. Then,

    (1) $s > t > c$ (2) $c > t > s$ (3) $c > s > t$ (4) $s > c > t$

43. In the figure (not drawn to scale) given below, P is a point on AB such that AP : PB = 4 : 3. PQ is parallel to AC and QD is parallel to CP. In triangle ARC, angle ARC = 90°, and in triangle PQS, angle PSQ = 90°. The length of QS is 6 cms. What is ratio AP : PD?

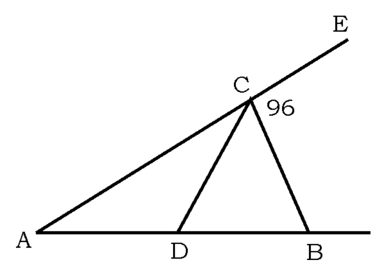


    (1) 10 : 3 (2) 2 : 1 (3) 7 : 3 (4) 8 : 3

44. A car is being driven, in a straight line and at a uniform speed, towards the base of a vertical tower. The top of the tower is observed from the car and, in the process, it takes 10 minutes for the angle of elevation to change from 45° to 60°. After how much more time will this car reach the base of the tower?

    (1) $5(\sqrt{3} + 1)$ (2) $6(\sqrt{3} + \sqrt{2})$ (3) $7(\sqrt{3} - 1)$ (4) $8(\sqrt{3} - 2)$

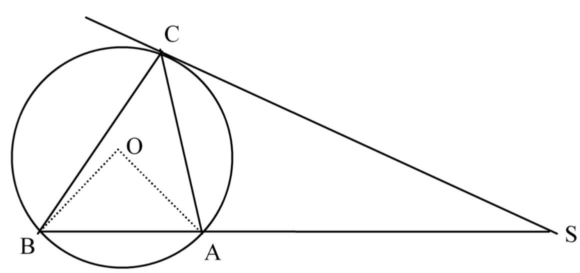
45. In the figure (not drawn to scale) given below, if AD = CD = BC, and angle BCE = 96°, how much is angle DBC?



    (1) 32° (2) 84°

    (3) 64° (4) Cannot be determined.

46. In the figure given below (not drawn to scale), A, B and C are three points on a circle with centre O. The chord BA is extended to a point S such that CS becomes a tangent to the circle at point C. If angle ASC = 30° and angle ACS = 50°, then the angle angle BOA is:



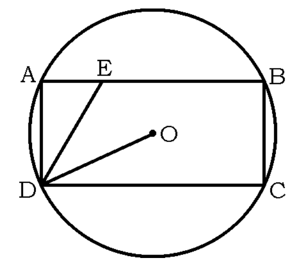
    (1) 100° (2) 150°

    (3) 80° (4) Not possible to determine

47. Let S_1 be a square of side a. Another square S_2 is formed by joining the mid-points of the sides of S_1. The same process is applied to S_2 to form yet another square S_3, and so on. If A_1, A_2, A_3, ... be the areas and P_1, P_2, P_3, ... be the perimeters of S_1, S_2, S_3, ... respectively, then the ratio $\frac{P_1 + P_2 + P_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}$ equal :

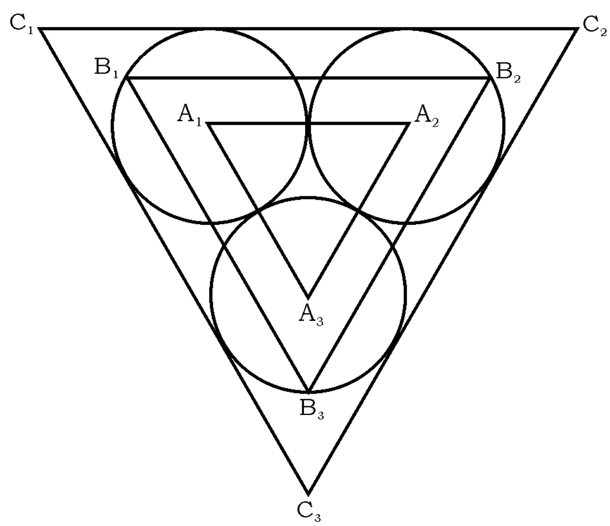
(1) $\frac{2(1+\sqrt{2})}{a}$ (2) $\frac{2(2-\sqrt{2})}{a}$ (3) $\frac{2(2+\sqrt{2})}{a}$ (4) $\frac{2(1+2\sqrt{2})}{a}$

48. In the figure below (not drawn to scale), rectangle ABCD is inscribed in the circle with center at O. The length of side AB is greater than that of side BC. The ratio of the area of the circle to the area of the rectangle ABCD is $\pi : \sqrt{3}$. The line segment DE intersects AB at E such that angle ODC = angle ADE. What is the ratio AE: AD?



(1) 1 : √3 (2) 2 : √2 (3) 2 : √3 (4) 1 : 2

Directions for questions 49 to 51: Consider three circular parks of equal size with centers at A_1, A_2, and A_3 respectively. The parks touch each other at the edge as shown in the figure (not drawn to scale). There are three paths formed by the triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$, as shown. Three sprinters A, B, and C begin running from points A_1, B_1, and C_1 respectively. Each sprinter traverses her respective triangular path clockwise and returns to her starting point.



49. Let the radius of each circular park be r, and the distances to be traversed by the sprinters A, B and C be a, b and c, respectively. Which of the following is true?

(1) $b - a = c - b = 3\sqrt{3}r$ (2) $b - a = c - b = \sqrt{3}r$

(3) $b = \frac{a+c}{2} = 2(1+\sqrt{3})r$ (4) $c = 2b - a = (2+\sqrt{3})r$

50. Sprinter A traverses distances A_1A_2, A_2A_3, and A_3A_1 at average speeds of 20, 30, and 15, respectively. B traverses her entire path at a uniform speed of $10\sqrt{3}+20$. C traverses distances C_1C_2, C_2C_3, and C_3C_1 at average speeds of $\frac{40}{3}(\sqrt{3}+1), \frac{40}{3}(\sqrt{3}+1)$, and 120, respectively. All speeds are in the same unit. Where would B and C be respectively when A finishes her sprint?

(1) B_1, C_1 (2) B_3, C_3

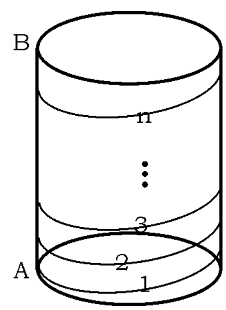
(3) B_1, C_3 (4) B_1, Somewhere between C_3 and C_1

51. Sprinters A, B and C traverse their respective paths at uniform speeds of u, v and w respectively. It is known that $u^2 : v^2 : w^2$ is equal to Area A : Area B : Area C, where Area A, Area B and Area C are the areas of triangles $A_1A_2A_3$, $B_1B_2B_3$, and $C_1C_2C_3$, respectively. Where would A and C be when B reaches point B_3?

(1) A_2, C_3 (2) A_3, C_3

(3) A_3, C_2 (4) Somewhere between A_2 and A_3, Somewhere between C_3 and C_1.

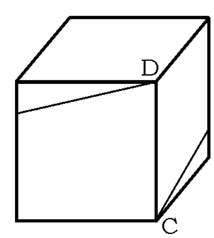
Directions for questions 52 to 54: Answer the questions on the basis of the information given below. Consider a cylinder of height h cms and radius $r = \frac{2}{\pi}$ cms as shown in the figure (not drawn to scale). A string of a certain length, when wound on its cylindrical surface, starting at point A and ending at point B, gives a maximum of n turns (in other words, the string's length is the minimum length required to wind n turns).



52. What is the vertical spacing in cms between two consecutive turns?

(1) $\frac{h}{n}$ (2) $\frac{h}{\sqrt{n}}$ (3) $\frac{h}{n^2}$ (4) Cannot be determined

53. The same string, when wound on the exterior four walls of a cube of side n cms, starting at point C and ending at point D, can give exactly one turn (see figure, not drawn to scale). The length of the string, in cms, is



(1) $\sqrt{2}\,n$ (2) $\sqrt{17}\,n$ (3) n (4) $\sqrt{13}\,n$

54. In the setup of the previous two questions, how is h related to n?

(1) $h = \sqrt{2}\,n$ (2) $h = \sqrt{17}\,n$ (3) $h = n$ (4) $h = \sqrt{13}\,n$

55. A piece of paper is in the shape of a right angled triangle and is cut along a line that is parallel to the hypotenuse, leaving a smaller triangle. There was a 35% reduction in the length of the hypotenuse of the triangle. If the area of the original triangle was 34 square inches before the cut, what is the area (in square inches) of the smaller triangle?

(1) 16.665 (2) 16.565 (3) 15.465 (4) 14.365

56. A square tin sheet of side 12 inches is converted into a box with open top in the following steps: The sheet is placed horizontally; Then, equal sized squares, each of side x inches, are cut from the four corners of the sheet; Finally, the four resulting sides are bent vertically upwards in the shape of a box. If x is an integer, then what value of x maximizes the volume of the box?

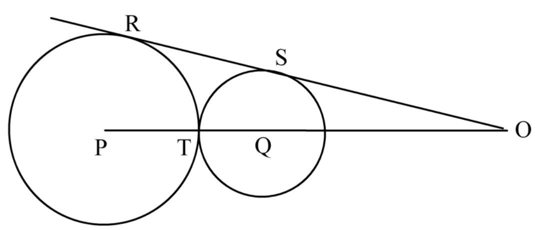
(1) 3 (2) 4 (3) 1 (4) 2

CAT 2004

57. A father and his son are waiting at a bus stop in the evening. There is a lamp post behind them. The lamp post, the father and his son stand on the same straight line. The father observes that the shadows of his head and his son's head are incident at the same point on the ground. If the heights of the lamp post, the father and his son are 6 metres, 1.8 metres and 0.9 metres respectively, and the father is standing 2.1 metres away from the post, then how far (in metres) is the son standing from his father?

(1) 0. 9 (2) 0.75 3. 0.6 (4) 0.45

Directions for questions 58 to 60: In the adjoining figure, I and II are circles with centres P and Q respectively. The two circles touch each other and have a common tangent that touches them at points R and S respectively. This common tangent meets the line joining P and Q at O. The diameters of I and II are in the ratio 4 : 3. It is also known that the length of PO is 28 cm.



58. What is the ratio of the length of PQ to that of QO?

(1) 1 : 4 (2) 1 : 3 (3) 3 : 8 (4) 3 : 4

59. What is the radius of the circle II?

(1) 2 cm (2) 3 cm (3) 4 cm (4) 5cm

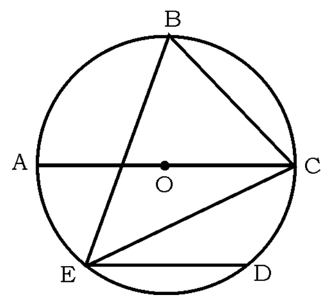
60. The length of SO is

(1) $8\sqrt{3}$ cm (2) $10\sqrt{3}$ cm (3) $12\sqrt{3}$ cm (4) $14\sqrt{3}$ cm

61. A rectangular sheet of paper, when halved by folding it at the mid point of its longer side, results in a rectangle, whose longer and shorter sides are in the same proportion as the longer and shorter sides of the original rectangle. If the shorter side of the original rectangle is 2, what is the area of the smaller rectangle?

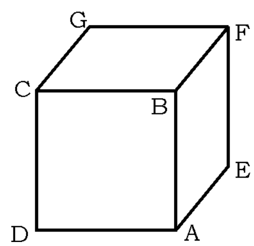
(1) $4\sqrt{2}$ (2) $2\sqrt{2}$ (3) $\sqrt{2}$ (4) None of these

62. In the adjoining figure, chord ED is parallel to the diameter AC of the circle. If angle CBE = 65°, then what is the value of angle DEC?



(1) 35° (2) 55° (3) 45° (4) 25°

63. If the lengths of diagonals DF, AG and CE of the cube shown in the adjoining figure are equal to the three sides of a triangle, then the radius of the circle circumscribing that triangle will be



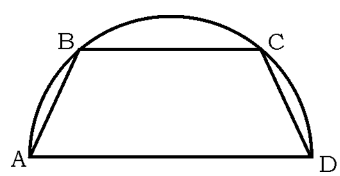
(1) equal to the side of the cube (2) $\sqrt{3}$ times the side of the cube

(3) $1/\sqrt{3}$ times the side of the cube (4) impossible to find from the given information

64. Let C be a circle with centre P_0 and AB be a diameter of C. Suppose P_1 is the mid point of the line segment P_0B, P_2 is the mid point of the line segment P_1B and so on. Let C_1, C_2, C_3, ... be circles with diameters P_0P_1, P_1P_2, P_2P_3 ... respectively. Suppose the circles C_1, C_2, C_3, are all shaded. The ratio of the area of the unshaded portion of C to that of the original circle C is

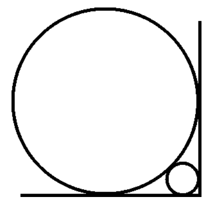
(1) 8 : 9 2. 9 : 10 (3) 10 : 11 (4) 11 : 12

65. On a semicircle with diameter AD, chord BC is parallel to the diameter. Further, each of the chords AB and CD has length 2, while AD has length 8. What is the length of BC?



(1) 7.5 (2) 7 (3) 7.75 (4) None of the above

66. A circle with radius 2 is placed against a right angle. Another smaller circle is also placed as shown in the adjoining figure. What is the radius of the smaller circle?



(1) $3 - 2\sqrt{2}$ (2) $4 - 2\sqrt{2}$ 3. $7 - 4\sqrt{2}$ (4) $6 - 4\sqrt{2}$

CAT 2005

67. Two identical circles intersect so that their centres, and the points at which they intersect, form a square of side 1 cm. The area in sq. cm of the portion that is common to the two circles is

(1) $\frac{\pi}{4}$ (2) $\frac{\pi}{2} - 1$ (3) $\frac{\pi}{5}$ (4) $\sqrt{2} - 1$

68. A jogging park has two identical circular tracks touching each other, and a rectangular track enclosing the two circles. The edges of the rectangles are tangential to the circles. Two friends, A and B, start jogging simultaneously from the point where one of the circular tracks touches the smaller side of the rectangular track. A jogs along the rectangular track, while B jogs along the two circular tracks in a figure of eight. Approximately, how much faster than A does B have to run, so that they take the same time to return to their starting point?

(1) 3.88% (2) 4.22% (3) 4.44% (4) 4.72%

69. What is the distance in cm between two parallel chords of lengths 32 cm and 24 cm in a circle of radius 20 cm?

(1) 1 or 7 (2) 2 or 14 (3) 3 or 21 (4) 4 or 28

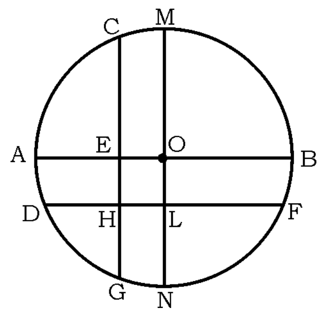
70. Four points A, B, C, and D lie on a straight line in the X-Y plane, such that AB = BC = CD, and the length of AB is 1 metre. An ant at A wants to reach a sugar particle at D. But there are insect repellents kept at points B and C. The ant would not go within one metre of any insect repellent. The minimum distance in metres the ant must traverse to reach the sugar particle is

(1) $3\sqrt{2}$ (2) $1 + \pi$ (3) $\frac{4\pi}{3}$ (4) 5

71. Rectangular tiles each of size 70 cm by 30 cm must be laid horizontally on a rectangular floor of size 110 cm by 130 cm, such that the tiles do not overlap. A tile can be placed in any orientation so long as its edges are parallel to the edges of the floor. No tile should overshoot any edge of the floor. The maximum number of tiles that can be accommodated on the floor is

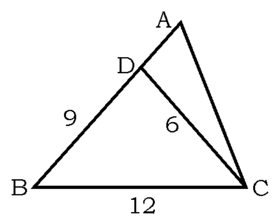
(1) 4 (2) 5 (3) 6 (4) 7

72. In the following figure, the diameter of the circle is 3 cm. AB and MN are two diameters such that MN is perpendicular to AB. In addition, CG is perpendicular to AB such that AE : EB = 1 : 2, and DF is perpendicular to MN such that NL : LM = 1 : 2. The length of DH in cm is



(1) $2\sqrt{2}-1$ (2) $\frac{2\sqrt{2}-1}{2}$ (3) $\frac{3\sqrt{2}-1}{2}$ (4) $\frac{2\sqrt{2}-1}{3}$

73. Consider the triangle ABC shown in the following figure where BC = 12 cm, DB = 9 cm, CD = 6 cm and angle BCD = angle BAC. What is the ratio of the perimeter of the triangle ADC to that of the triangle BDC?



(1) 7/9 (2) 8/9 (3) 6/9 (4) 5/9

74. P, Q, S, and R are points on the circumference of a circle of radius *r*, such that PQR is an equilateral triangle and PS is a diameter of the circle. What is the perimeter of the quadrilateral PQSR?

(1) $2r\left(1+\sqrt{3}\right)$ (2) $2r\left(2+\sqrt{3}\right)$ (3) $r\left(1+\sqrt{5}\right)$ (4) $2r+\sqrt{3}$

75. A rectangular floor is fully covered with square tiles of identical size. The tiles on the edges are white and the tiles in the interior are red. The number of white tiles is the same as the number of red tiles. A possible value of the number of tiles along one edge of the floor is

(1) 10 (2) 12 (3) 14 (4) 16

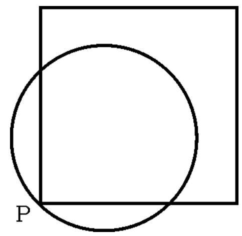
CAT 2006:

76. The length, breadth and height of a room are in the ratio 3 : 2 : 1. If the breadth and height are halved while the length is doubled, then the total area of the four walls of the room will

(1) remain the same (2) decrease by 13.64% (3) decrease by 15%

(4) decrease by 18.75% (5) decrease by 30%

Directions for questions 77 and 78: A punching machine is used to punch a circular hole of diameter two units from a square sheet of aluminum of width 2 units, as shown below. The hole is punched such that circular hole touches one corner P of the square sheet and the diameter of the hole originating at P is in line with a diagonal of the square.



77. The proportion of the sheet area that remains after punching is:

    (1) $(\pi-2)/8$ (2) $(6-\pi)/8$ (3) $(4-\pi)/4$ (4) $(\pi-2)/4$ (5) $(14-3\pi)/6$

78. Find the area of the part of the circle (round punch) falling outside the square sheet.

    (1) $\pi/4$ (2) $(\pi-1)/2$ (3) $(\pi-1)/4$ (4) $(\pi-2)/2$ (5) $(\pi-2)/4$

79. A semicircle is drawn with AB as its diameter. From C, a point on AB, a line perpendicular to AB is drawn meeting the circumference of the semicircle at D. Given that AC = 2 cm and CD = 6 cm, the area of the semicircle (in sq. cm.) will be:

    (1) 32π (2) 50π (3) 40.5π (4) 81π (5) Undeterminable

80. An equilateral triangle BPC is drawn inside a square ABCD. What is the value of the angle APD in degrees?

    (1) 75 (2) 90 (3) 120 (4) 135 (5) 150

CAT 2007

Directions for questions 81 & 82: The questions are followed by two statements A and B. Indicate your responses based on the following directives:

Mark (1) if the question can be answered using A alone but not using B alone.

Mark (2) if the question can be answered using B alone but not using A alone.

Mark (3) if the question can be answered using A and B together, but not using either A or B alone.

Mark (4) if the question cannot be answered even using A and B together.

81. Rahim plans to draw a square JKLM with a point O on the side JK but is not successful. Why is Rahim unable to draw the square?

    A: The length of OM is twice that of OL.

    B: The length of OM is 4 cm.

82. ABC Corporation is required to maintain at least 400 kilolitres of water at all times in its factory, in order to meet safety and regulatory requirements. ABC is considering the suitability of a spherical tank with uniform wall thickness for the purpose. The outer diameter of the tank is 10 meters. Is the tank capacity adequate to meet ABC's requirements?

    A: The inner diameter of the tank is at least 8 meters.

    B: The tank weighs 30,000 kg when empty, and is made of a material with density of 3 gm/cc.

83. Two circles with centres P and Q cut each other at two distinct point A and B. The circles have the same radii and neither P nor Q falls within the intersection of the circles. What is the smallest range that includes all possible values of the angle AQP in degrees?

(1) Between 0 and 30 (2) Between 0 and 60 (3) Between 0 and 75

(4) Between 0 and 45 (5) Between 0 and 90

CAT 2008

84. In a triangle ABC, the lengths of the sides AB and AC equal 17.5 cm and 9 cm respectively. Let D be a point on the line segment BC such that AD is perpendicular to BC. If AD = 3 cm, then what is the radius (in cm) of the circle circumscribing the triangle ABC?

(1) 17.05 (2) 27.85 (3) 22.45 (4) 32.25 (5) 26.25

85. Consider obtuse-angled triangles with sides 8 cm, 15 cm and x cm. If x is an integer, then how many such triangles exist?

(1) 5 (2) 21 (3) 10 (4) 15 (5) 14

86. Consider a square ABCD with midpoints E, F, G, H of AB, BC, CD and DA respectively. Let L denote the line passing through F and H. Consider points P and Q, on L and inside ABCD, such that the angles APD and BQC both equal 120°. What is the ratio of the area of ABQCDP to the remaining area inside ABCD?

(1) $\frac{4\sqrt{2}}{3}$ (2) $2+\sqrt{3}$ (3) $\frac{10-3\sqrt{3}}{9}$ (4) $1+\frac{1}{\sqrt{3}}$ (5) $2\sqrt{3}-1$

87. Two circles, both of radii 1 cm, intersect such that the circumference of each one passes through the centre of the circle of the other. What is the area (in sq cm) of the intersecting region?

(1) $\frac{\pi}{3}-\frac{\sqrt{3}}{4}$ (2) $\frac{2\pi}{3}+\frac{\sqrt{3}}{2}$ (3) $\frac{4\pi}{3}-\frac{\sqrt{3}}{2}$ (4) $\frac{4\pi}{3}+\frac{\sqrt{3}}{2}$ (5) $\frac{2\pi}{3}-\frac{\sqrt{3}}{2}$

88. Consider a right circular cone of base radius 4 cm and height 10 cm. A cylinder is to be placed inside the cone with one of the flat surface resting on the base of the cone. Find the largest possible total surface area (in sq. cm) of the cylinder.

(1) $\frac{100\pi}{3}$ (2) $\frac{80\pi}{3}$ (3) $\frac{120\pi}{7}$ (4) $\frac{130\pi}{9}$ (5) $\frac{130\pi}{9}$

Answer Key

Exercise 1: Angles in a Triangle

1. 1	2. 1	3. 3	4. 2	5. 1
6. 4	7. 2	8. 1		

Exercise 2: Sides of a Triangle

1. 2	2. 4	3. 4	4. 2	5. 3
6. 2	7. 1	8. 2		

Exercise 3: Relation between angle and sides

1. 3	2. 1	3. 4	4. 3	5. 2
6. 4				

Exercise 4: Area of triangle & Basic Proportionality Theorem

1. 3	2. 1	3. 1	4. 4	5. 3
6. 2	7. 3	8. 4		

Exercise 5: 4 Lines and 4 Points of Triangles

1. 3	2. 2	3. 2	4. 3	5. 1
6. 3	7. 3	8. 4		

Exercise 6: Right Angle Triangles

1. 4	2. 2	3. 1	4. 3	5. 2
6. 1	7. 2	8. 4	9. 2	10. 3

Exercise 7: Similarity of triangles

1. 1	2. 2	3. 4	4. 2	5. 3
6. 1	7. 3	8. 4	9. 1	10. 2

Exercise 8: Quadrilaterals and Polygons

1. 3	2. 2	3. 4	4. 2	5. 4
6. 2	7. 3	8. 2	9. 3	10. 1
11. 1	12. 3			

Exercise 9: Circles

1. 2	2. 1	3. 1	4. 3	5. 2
6. 1	7. 2	8. 4	9. 4	10. 4
11. 3	12. 1	13. 3	14. 2	15. 2

Exercise 10: Solids

1. 4	2. 2	3. 3	4. 1	5. 3
6. 2	7. 4	8. 3	9. 1	10. 3

CAT Questions

1. 3	2. 3	3. 1	4. 4	5. 4
6. 3	7. 1	8. 1	9. 4	10. 3
11. 2	12. 1	13. 2	14. 1	15. 3
16. 2	17. 3	18. 4	19. 2	20. 1
21. 1	22. 2	23. 2	24. 3	25. 4
26. 4	27. 3	28. 3	29. 4	30. 1
31. 2	32. 3	33. 3	34. 2	35. 1
36. 2	37. 4	38. 2	39. 1	40. 3
41. 2	42. 3	43. 3	44. 1	45. 3
46. 1	47. 3	48. 1	49. 2	50. 3
51. 2	52. 1	53. 2	54. 3	55. 4
56. 1	57. 4	58. 2	59. 2	60. 3
61. 2	62. 4	63. 1	64. 4	65. 2
66. 4	67. 2	68. 4	69. 4	70. 2
71. 3	72. 2	73. 1	74. 1	75. 2
76. 5	77. 2	78. 4	79. 2	80. 5
81. 1	82. 2	83. 2	84. 5	85. 3
86. 5	87. 5	88. 1		

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