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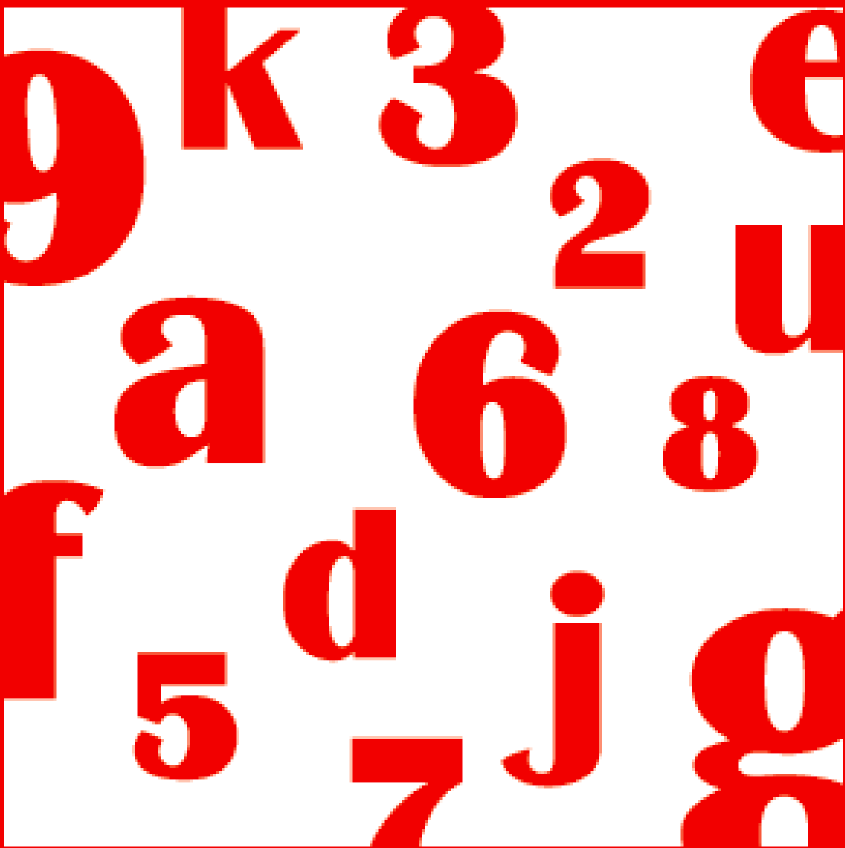

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Index

Ratio, Proportion & Variation

Ratios

Ratios are just comparisons – comparing the sizes, the numbers, the quantum of any two or more comparable quantities.

When we say that the ratio of the amount that A, B, C has is 2 : 1 : 3, we are comparing the amounts that they have. It is not the same as A, B and C having Rs. 2, Re 1 and Rs. 3 respectively. When we talk of ratios, we are essentially commenting on the 'relative' amounts each has (and not actual amounts) i.e. A has twice the amount that B has; C has thrice the amount that B has; and for every Rs. 2 that A has, C has Rs. 3.

Hence, they having Rs. 2, Re. 1 and Rs. 3 is just one possibility. They might also have Rs. 4, Rs. 2, Rs. 6 or for that matter also Rs. 20, Rs. 10 and Rs. 30. In fact there can be any number of possible amounts that they have, with the only constraint that relatively, the amounts should conform to 2 : 1 : 3.

Mathematically speaking, the ratio $a : b$ is same as $\frac{a}{b}$. However this interpretation can be used only while considering ratios of two quantities. The meaning of 'ratio of number of boys and girls in a class is 3 : 4' is $\frac{\text{number of boys}}{\text{number of girls}} = \frac{3}{4}$

Since $\frac{3 \times \cancel{2}}{4 \times \cancel{2}}$ is same as $\frac{3}{4}$, if the number of boys is 3 × 2, then the number of girls has to be 4 × 2.

Similarly, since $\frac{3 \times \cancel{3}}{4 \times \cancel{3}}$ is same as $\frac{3}{4}$, if the number of boys is 3 × 3, then the number of girls has to be 4 × 3.

And so on. Or in general since $\frac{3 \times \cancel{k}}{4 \times \cancel{k}}$ is same as $\frac{3}{4}$, if the number of boys is $3 \times k$, then the number of girls has to be $4 \times k$.

Using the same logic as above, if the ratio of the amount that A, B, C has is 2 : 1 : 3, the actual amounts that each of them has can be assumed as $2k$, k and $3k$ respectively.

E.g. 1: If the ages of a husband, wife and their child are in the ratio 13 : 11 : 3 and the average age of the family of the three is 36 years, find the difference between the age of the husband and the wife?

The ages of the husband, wife and the child can be assumed as $13k$, $11k$ and $3k$.

Since the average age of the family is 36 years, $\frac{13k + 11k + 3k}{3} = 36 \Rightarrow$

$27k = 108$ i.e. $k = 4$.

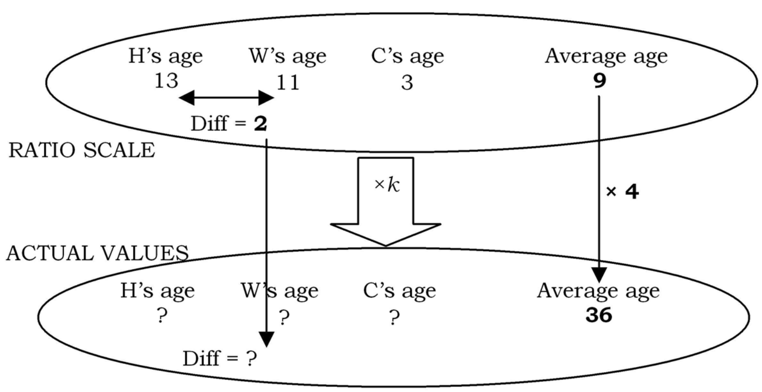
Thus, the ages of the husband and wife are 13×4 and 11×4 i.e. 52 and 44 and the required difference is 8 years.

Ratio scale & Actual values - Approach to use ratios to find actual values

To reduce the pencil work involved, we suggest the following pictorial understanding ...

Consider the underlying quantities, ages in above example, in two different scales – one the actual values (to be found) and one on a reduced scale, called ratio scale.

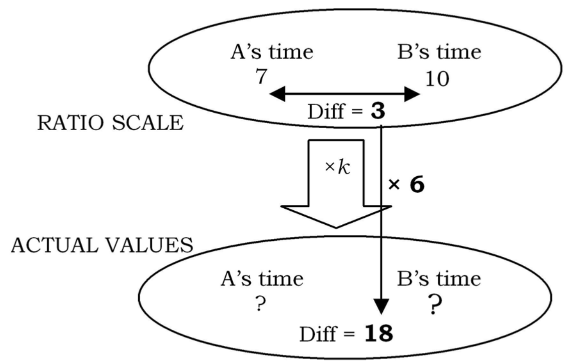
From the actual values, the common factor, k, is cancelled to arrive at the ratio scale. Thus, the path to find any actual value from corresponding ratio scale is to multiply by k



9 of ratio scale corresponds to 36 of actual value. Thus, to move from ratio scale to actual value one has to multiply by 4. We need the difference between husband's and wife's age. In ratio scale it is 2 and hence in actual values it will be 8.

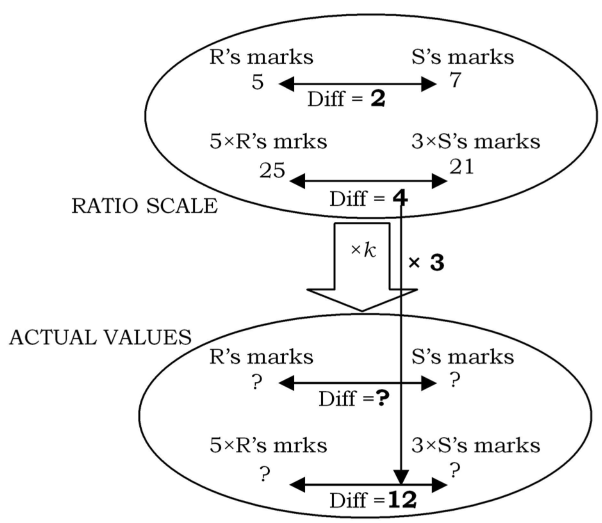
E.g. 2: The time taken by A and B to reach their office from home is in the ratio 7 : 10. If B takes 18 minutes more than A to reach office, find the time taken by each of them to reach office.

B takes 18 minutes more than A means that the difference in their time is 18 minutes.



Difference in time taken in the ratio scale = 10 – 7 = 3, corresponds to a difference in actual time of 18 minutes. Thus, all values of ratio scale have to be multiplied by 6 to arrive at actual values. Required times are 42 & 60.

E.g. 3: The ratio of the marks obtained by Ram and Shyam in their exams is 5 : 7. If five times the marks obtained by Ram is 12 more than thrice the marks obtained by Shyam, find the difference in their marks.



While the above picture seems a lot, it is as simple as:

4 of ratio scale corresponds to actual value of 12 and hence 2 of ratio scale will correspond to how much? And the answer is 6.

The above approach helps us in eliminating finding non-essentials like actual marks scored by Ram and Shyam and quickly arrive at the answer without the need to write equations with k.

E.g. 4: Divide Rs. 117 among A, B and C such that their shares are in the ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$ respectively.

It is going to be a bit cumbersome to assume the individual shares as $\frac{1}{2}k, \frac{1}{3}k, \frac{1}{4}k$. An alternative is to manipulate the given ratio such that we deal in natural numbers.

Multiplying all numbers of a ratio by same number leaves the ratio unchanged

It is very obvious that each of 30 : 40 or 300 : 400 or 60 : 80 is same as 3 : 4. Thus, multiplying each number in a ratio by the same number leaves the ratio unchanged. When we multiply each number in a ratio, in effect we are just increasing the magnitude of all the numbers and since it is the same number, relatively, the numbers are still same in comparison.

We can multiply all three with a same number such that the fractions turn out to be natural numbers. Needless to explain, the number has to be a multiple of 2, 3 and 4 and 12 would be a right choice in this case. The given ratio is same as $\frac{1}{2}\times 12 : \frac{1}{3}\times 12 : \frac{1}{4}\times 12$ i.e. 6 : 4 : 3.

Now, 117 is the sum of the three shares i.e. the sum in ratio scale, 13, corresponds to 117. So $k = 9$.

The individual shares are 6 × 9, 4 × 9, 3 × 9 i.e. 54, 36 and 27.

E.g. 5: Divide Rs. 117 among A, B and C such that $\frac{\text{A's share}}{2} = \frac{\text{B's share}}{3} = \frac{\text{C's share}}{4}$

Assuming $\frac{\text{A's share}}{2} = \frac{\text{B's share}}{3} = \frac{\text{C's share}}{4} = k$, we have A’s share = $2k$, B’s share = $3k$ and C’s share = $4k$. Thus, the share of A, B and C are in the ratio 2 : 3 : 4. (this ratio is different from that obtained in earlier example namely 6 : 4 : 3)

The sum of the share, in the ratio scale, 9, corresponds to 117 i.e. $k = 13$. So the individual shares are 2 × 13, 3 × 13, 4 × 13 i.e. 26, 39 and 52.

Don't confuse ...

... between "A, B and C's shares are in the ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$" and the data that

"$\frac{\text{A's share}}{2} = \frac{\text{B's share}}{3} = \frac{\text{C's share}}{4}$".

In the former case, since ratios are given in fractions, $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$, we multiply with the LCM of denominators to get rid of the fractions. Thus, A, B and C's share are in ratio 6 : 4 : 3.

In the latter case quite a few students see the 2, 3 and 4 in the denominator and hence again think of multiplying with the LCM, which is a misleading path. One just needs to equate $\frac{\text{A's share}}{2} = \frac{\text{B's share}}{3} = \frac{\text{C's share}}{4}$ to k and one gets A, B and C's share are in ratio 2 : 3 : 4.

In fact the data "2 × A'share = 3 × B's share = 4 × C's share" will result in A, B and C's share being in ratio $\frac{1}{2}:\frac{1}{3}:\frac{1}{4}$ i.e. 6 : 4 : 3.

Thus, $\frac{\text{A's share}}{x} = \frac{\text{B's share}}{y} = \frac{\text{C's share}}{z}$ means the shares are in ratio $x : y : z$

And x × A's share = y × B's share = z × C's share means the shares are in ratio $\frac{1}{x}:\frac{1}{y}:\frac{1}{z}$

Given $a : b$ and $b : c$, finding $a : c$

In such cases, the first ratio gives us the relation between a and b; the second ratio gives us the relation between b and c; thus logic says that we should be able to find the relation between a and c because <u>b is common to the two relation</u>. And b being common to the two relations is exactly what we use to find the answer, as done in the following examples.

E.g. 6: If $a : b$ is 3 : 4 and $b : c$ is 5 : 6, find the ratio $a : b : c$.

Since b is common to the two ratios, we should make the numeric value of b the same in both the ratios. Also remember that when all terms of a ratio are multiplied with the same constant, the ratio does not change.

Thus in the first ratio b could be changed to any multiple of 4 and in the second ratio b could be changed to any multiple of 5. Thus, we should make b a multiple of 4 and 5 i.e. 20.

$a : b$ is 3 × 5 : 4 × 5 i.e. 15 : 20

$b : c$ is 5 × 4 : 6 × 4 i.e. 20 : 24

Thus when b is 20, a is 15 and c is 24 and the required ratio of $a : b : c$ is 15 : 20 : 24.

E.g. 7: If $a : b$ is 2 : 5, $b : c$ is 3 : 5 and $c : d$ is 5 : 4, find the ratio $a : b : c : d$.

In the ratios of $a : b$ and $b : c$, if b has to be the same numeric value, it should be a multiple of 5 and 3 i.e. 15. Thus,

$a : b$ is 2 : 5 which is same as 6 : 15

$b : c$ is 3 : 5 which is same as 15 : 25.

Thus $a : b : c$ is 6 : 15 : 25

Now c is common to $a : b : c$ and $c : d$ and thus making c equal in the two ratios,

$c : d$ is 5 : 4 which is same as 25 : 20.

Thus $a : b : c : d$ is 6 : 15 : 25 : 20

In the above question had the question been finding just $a : d$, one could have avoided all this work and just done the following: $\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{2}{5} \times \frac{3}{5} \times \frac{5}{4} = \frac{3}{10}$

Please be careful of which term to be made numerically equal. It should be that term which appears in both the relations.

E.g. 8: Given $a : b$ is 6 : 5 and $a : c$ is 15 : 11, find the ratio $a : b : c$.

In this example, we would make a equal in the two ratios as it is common to both the relations. Now, a could be any multiple of 6 in first ratio and any multiple of 15 in second ratio, thus let's make a equal to 30

$a : b$ is 6 : 5 i.e. 30 : 25

$a : c$ is 15 : 11 i.e. 30 : 22

Thus $a : b : c$ is 30 : 25 : 22

Standard Questions on Ratios

E.g. 9: The ratio of the present ages of a father and a son is 8 : 1. Eight years hence, the ratio would be 10 : 3. What is the present age of the father?

Assuming the present ages of the father and son to be $8k$ and k respectively, the ages after 8 years will be $8k + 8$ and $k + 8$. Thus we have,

$\frac{8k+8}{k+8} = \frac{10}{3}$ i.e. $24k + 24 = 10k + 80$ i.e. $14k = 56$ i.e. $k = 4$.

Thus present age of father is 8 × 4 = 32 years.

E.g. 10: If the incomes of A and B are in ratio 6 : 5 and their expenses are in ratio 5 : 4, find the ratio of their savings.

Error-prone area

When two ratios are given, the constants need to be taken as different

Quite a few students assume the income as 600 and 500 and the expenses as 500 and 400 and find the savings as 100 in each case and thus, conclude the answer to be 1 : 1.

Few others, approach it mathematically and yet arrive at the same erroneous answer. The incomes are assumes as $6k$ and $5k$, expenses are assumed as $5k$ and $4k$ and the ratio of savings is found as $(6k – 5k) : (5k – 4k)$ i.e. $k : k$ i.e. 1 : 1.

The above answer is wrong as evident from the data that the incomes are 600 and 500 (in the ratio 6 : 5) and the expenses are 50 and 40 (in the ratio 5 : 4). In this case, their savings are 550 and 460 which are not in the ratio 1 : 1.

There are two ratios, ratio of income 6 : 5 and ratio of expenses 5 : 4. And the two ratios are independent of each other. Thus, if the income is assumed as $6k$ and $5k$, i.e. some multiple of 6 : 5, the expenses need not be the same multiple of 5 : 4. Thus, the multiple of 5 : 4, that needs to be used has to be different from the earlier multiple, say $5n$ and $4n$.

Assuming the incomes as $6k$ and $5k$ and the expenses as $5n$ and $4n$, we get the ratio of savings as $\frac{6k-5n}{5k-4n}$, the value of which will keep changing based on the relative values of k and n. Thus, there is no unique answer to the ratio of savings, as must be evident from the two specific cases discussed in the box above – with incomes as 600 & 500 and expenses as 500 & 400, the ratio of savings is 1 : 1; and with incomes as 600 & 500 and expenses as 50 & 40, the ratio of savings is 55 : 46.

What if both ratio of income and ratio of savings is 6 : 5 and 6 : 5?

Approaching the problem mathematically, assuming incomes as $6k$ and $5k$ and expenses as $6n$ and $5n$, the ratio of savings is $\frac{6k-6n}{5k-5n} = \frac{6(k-n)}{5(k-n)} = \frac{6}{5}$. (The above is true when each of them save something i.e. except the case when income = expenditure i.e. when except when $k = n$. In this case, the savings of each will be 0)

Thus, we see that in case both ratio of income and ratio of expenses are in ratio $a : b$, then savings will also be in the ratio $a : b$. And in case the ratio of income and expenses is different, the ratio of savings cannot be determined.

E.g. 11: The ratio of the income and expenditure of Amit and Sumit are 5 : 8 and 11 : 10. If Amit and Sumit save Rs. 7900 and Rs. 21000 respectively, find their incomes.

We have already learnt that two different constant of proportionality for the two ratios will have to be used.

Thus assuming the income of Amit and Sumit as $5k$ and $8k$ and assuming the expenditure of the two as $11n$ and $10n$, we have the following two equations: $5k - 11n = 7900$ and $8k - 10n = 21000$

i.e. $50k - 110n = 79000$ and $88k - 110n = 231000$

Subtracting, $38k = 152000$ i.e. $k = 4000$.

Thus the incomes of Amit and Sumit are 5 × 4000 = 20,000 and 8 × 4000 = 32,000 respectively.

Exercise

1. If $a : b$ is 3 : 4, find the ratio $(7a - 4b) : (3a + b)$.

   1. $\frac{16}{15}$ 2. $\frac{5}{13}$ 3. $-\frac{5}{13}$ 4. $-\frac{16}{15}$ 5. Cannot be determined

2. If $(7x - 4y) : (3x + y)$ is 5 : 13, find the ratio of x and y.

   1. $\frac{4}{3}$ 2. $\frac{3}{4}$ 3. $-\frac{4}{3}$ 4. $-\frac{3}{4}$ 5. Cannot be determined

3. If $a : b$ is 2 : 3, $b : c$ is 4 : 5 and $c : d$ is 6 : 7, find the ratio of $a : b : c : d$

   1. 2 : 3 : 4 : 5 2. 4 : 5 : 6 : 7 3. 2 : 3 : 6 : 7
   4. 2 : 12 : 30 : 7 5. 16 : 24 : 30 : 35

4. If $a : b$ is 3 : 4; $b : c$ is 5 : 3 and $a : d$ is 2 : 5, find the ratio of $c : d$.

   1. 3 : 5 2. 8 : 15 3. 1 : 5 4. 5 : 8 5. 3 : 8

5. Two friends separated by a certain distance start walking towards each other. When they meet one of them has walked 20 meters more than the other. If the ratio of the distances that each has covered is 2 : 3, find the distance that originally separated them.

   1. 20 2. 40 3. 60 4. 80 5. 100

6. Divide 2220 in the ratio $\frac{1}{4} : \frac{1}{5} : \frac{1}{6}$

   1. 900, 720, 600 2. 600, 720, 900 3. 592, 740, 888
   4. 888, 740, 592 5. None of these

7. If $3A = 4B = 5C = 6D$ and $A + B + C + D = 1026$, find the values of A, B, C and D.

   1. 171, 228, 285, 342 2. 342, 285, 228, 171 3. 180, 216, 270, 360
   4. 360, 270, 216, 180 5. None of these

8. All the 585 students of a school are divided into three different groups such that the half the number of students in first group, one-third the number of students in second group and one-fourth the number of students in third group are equal. Find the number of students in each group.

1. 130, 195, 260 2. 260, 195, 130 3. 270, 130, 135

4. 135, 180, 270 5. None of these

9. Present ages of Sameer and Anand are in the ratio 5 : 4. Three years hence, the ratio of their ages will become 11 : 9. What is Anand's present age?

1. 16 2. 20 3. 24 4. 28 5. 32

10. Six years back, the ratio of the ages of a husband and his wife was 6 : 5. Six years hence, their ages will be in the ratio 10 : 9. What is the ratio of their present ages.

1. 7 : 6 2. 8 : 7 3. 9 : 10 4. 18 : 15 5. 5 : 4

11. For which of the following cases can the ratio of expenditures of A and B be found?

I. The incomes of A and B are in ratio of 3 : 5 and savings of A and B are in ratio of 8 : 11

II. The incomes of A and B are in ratio of 3 : 5 and savings of A and B are in ratio of 1 : 3

III. The incomes of A and B are in ratio of 3 : 5 and savings of A and B are in ratio of 3 : 5

1. None 2. III 3. II and III 4. II 5. All three

12. The total income of A, B and C is Rs. 80,000. Their expenditure are Rs. 8,000, Rs. 12,000 and Rs. 15,000 respectively. If their savings are in the ratio of 2 : 3 : 4, find the income of B.

1. 5000 2. 12000 3. 18000 4. 27000 5. 35000

13. Two cylinders are such that the ratio of the radii of their base is 3 : 4 and the ratio of their heights are 4 : 3. Find the ratio of their volumes. (Volume of a cylinder = $\pi r^2 h$)

1. 3 : 4 2. 4 : 3 3. 1 : 1 4. 9 : 16 5. 27 : 64

14. The number of 1 Re, 50 paise and 25 paise coins in a bag is in the ratio 3 : 4 : 5. If the bag has Rs. 300, find the number of 50 paise coins.

1. 60 2. 96 3. 144 4. 192 5. 240

15. If $ab : bc : ca = 2 : 3 : 4$, find the ratio of $a^2 : b^2 : c^2$

1. 1 : 1 : 1 2. 4 : 9 : 16 3. 36 : 16 : 9 4. 6 : 4 : 3 5. 16 : 9 : 36

16. If $(a + b) : (b + c) : (c + a)$ is 3 : 4 : 5 and $a + b + c = 18$, find the value of $a \times b \times c$.

1. 60 2. 120 3. 144 4. 162 5. 172

Partnership

Partnership is when two or more people pool in money as capital for a common venture. The profit of the venture is then divided among the people depending on the amount of money that each has invested. Quite often the time duration for which an individual has invested money in the venture may also be different for different partners – one may quit the partnership withdrawing his investment prematurely or a new partner may join the venture sometimes later. If a partner quits, the venture does not cease to exist. After a pre-defined interval of time, the profits are distributed and the partner who exited early (or entered later) would consequently get a diminished share of profit as compared to that had he been there for complete duration.

The following lists out how the profit is divided in each of the different scenarios possible.

Different Investments, Same Time Period Of Investing

If the amount invested by the partners are C_1, C_2, C_3 then the profit is distributed in the ratio $C_1 : C_2 : C_3$.

E.g. 12: Rahul and Rohit get in Rs. 4000 and Rs. 5000 to fund a new venture. In what ratio should they divide the profit of Rs. 1,80,000 earned at the end of the year?

The profit has to be divided among Rahul and Rohit in the ratio of their investments i.e. 4 : 5

The total of the shares, in ratio scale, 9 corresponds to 1,80,000 i.e. $k = 20{,}000$

Thus Rahul's share = 4 × 20,000 = 80,000 and Rohit's share = 5 × 20,000 = 1,00,000.

When the investment and the time period is different

Let there be three partners, one invests C_1 for t_1 time, second invests C_2 for t_2 time and third invests C_3 for t_3 time. The profit is shared in the ratio $C_1 \times t_1 : C_2 \times t_2 : C_3 \times t_3$.

E.g. 13: A and B enter into a partnership with Rs. 8,000 and 15,000 respectively. After 3 months C joins them by investing Rs. 10,000. 4 months before the first year is completed, B quits, taking his invested amount back with him. In what ratio should the profit of Rs. 2,55,000 earned in the first year be distributed among the three?

A has invested 8,000 for 12 months; B has invested 15,000 for 8 months; C has invested 10,000 for 9 months.

Thus the profit has to be distributed in the ratio of
8 × 12 : 15 × 8 : 10 × 9 i.e. 16 : 20 : 15

The total of their shares, in ratio scale, 16 + 20 + 15 = 51 corresponds to 2,55,000 i.e. k = 5,000

A's share = 16 × 5,000 = 80,000; B's share = 20 × 5,000 = 1,00,000;
C's share = 15 × 5,000 = 75,000

Different Amounts Invested In Different Time Periods

Let's say in a partnership between A and B, A invests Rs. C_a for a time period of t_a. But B invests Rs. C_{b1} for a period of t_{b1} time and Rs. C_{b2} for a period of t_{b2} time. In this case the profit will be divided between A and B in the ratio
$C_a \times t_a : (C_{b1} \times t_{b1} + C_{b2} \times t_{b2})$

E.g. 14: A, B and C enter into a partnership. They invest Rs. 40,000, Rs. 80,000 and Rs. 1,20,000 respectively. At the end of the first year, B withdraws Rs. 40,000 while at the end of second year, C withdraws Rs. 80,000. In what ratio will the profit be shared at the end of three years?

The profit has to be shared in the ratio of
(40 × 3) : (80 × 1) + (40 × 2) : (120 × 2) + (40 × 1)

i.e. 120 : 160 : 280 i.e. 3 : 4 : 7.

Different ways of accounting for B's investment

One could have thought of B's investment in different ways, though all of them yield the same end-result ....

B invested 80,000 for 1 year and then 40,000 for next two years: Treated mathematically as 80 × 1 + 40 × 2 i.e. 160.

B invested 40,000 for the entire 3 year period and an additional 40,000 for the first year: Treated mathematically as 40 × 3 + 40 × 1 i.e. 160.

B invested 80,000 for entire 3 years less 40,000 for last two years: Treated mathematically as 80 × 3 – 40 × 2 i.e. 160.

Exercise

17. A starts a business with Rs.3,500 and after 5 months, B joins A. At the end of the year, the profit is divided between A and B in the ratio 2 : 3. What amount did B invest in the business?

1. 2800 2. 2700 3. 1800 4. 1400 5. 9000

18. Three partners shared the profit in a business in the ratio 5 : 7 : 8. They had partnered for 14 months, 8 months and 7 months respectively. What was the ratio of their investments.

1. 5 : 4 : 4 2. 20 : 49 : 56 3. 784 : 320 : 245

4. 19 : 15 : 15 5. None of these

19. A, B and C rent a pasture. A puts 10 oxen for 7 months, B puts 12 oxen for 5 months and C puts 15 oxen for 3 months for grazing. If the rent of the pasture is Rs. 175, find the share of C in the rent.

1. 45 2. 105 3. 30 4. 70 5. None of theses

20. A and B enter into a partnership by contributing Rs.1,00,000 and 1,50,000 respectively. After 3 months C joins them by contributing a capital of 2,00,000. Four months before the end of the year, B quits, taking his share of the capital away with him. Because of this A adds another 50,000 of capital from his side. At the end of the year the partnership makes a profit of Rs.1,65,000. What will be the difference between C's share and A's share of the profit?

1. 11,000 2. 13,000 3. 15,000 4. 8,250 5. 22,000

Proportion

When two ratio are equal, $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d are said to be in proportion.

Conversely, if a, b, c, d are in proportion, then we have $\frac{a}{b} = \frac{c}{d}$. The order is sacrosanct.

a & d are called the extremes and b & c are called the means.

From the above relation it should be obvious that if a, b, c and d are in proportion, then $a \times d = b \times c$ i.e. product of extremes = product of means.

Continued proportion

If a, b, c, d, e, ... are in continued proportion, then we have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \ldots\ldots$

In specific, if three numbers, a, b, c are in continued proportion, then $b^2 = a \times c$. In this case b is called the mean proportional to a & c.

Continued Proportion and Geometric Progression

If a, b, c, d, e, ... are in continued proportion, then we have $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e} = \ldots\ldots$ This relation can also be interpreted as the ratio of consecutive terms of the series, a, b, c, d, e, ... being equal. And you must be knowing (or will learn in algebra) that such a series is a Geometric Progression.

Thus, terms of any Geometric progression are in continued proportion. And consequently, the mean proportional to a & c is nothing but the Geometric Mean.

The above will be useful in certain situations when we have to assume three or four numbers that are in continued proportion. Rather than assume the numbers as a, b, c, d (which involves four variables), we can assume the numbers as a, ar, ar^2, ar^3 (reducing the variables involved to just 2).

Operations on Proportions

Componendo: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{b} = \frac{c+d}{d}$

Dividendo: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a-b}{b} = \frac{c-d}{d}$

Componendo & Dividendo: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Logic

The relation found by componendo can be proved by adding 1 to both sides of the proportion.

If $\frac{a}{b}=\frac{c}{d}$ is given. Adding 1 to both sides, $\frac{a}{b}+1=\frac{c}{d}+1$ i.e. $\frac{a+b}{b}=\frac{c+d}{d}$.

Similalry the relation found by dividendo can be proved by subtracting 1 from both sides of the proportion.

And taking the ratio of the results of componendo and that of dividendo will result in the relation found by componendo & dividendo.

The above have very limited usage and those limited scenarios can also be solved further without the use of the above.

Application

I. When fractions of the type $\frac{x+y}{x-y}$ are involved, performing componendo & dividendo on this fraction directly yields $\frac{x}{y}$. And we will find such fractions in couple of scenarios (boats & streams, relative speed in case of opposite & same direction, etc)

Thus, if $\frac{3x-4y}{3x+4y}=\frac{7}{2}$, rather than cross-multiplying & re-arranging, once just perform componendo & dividendo to result in $\frac{3x}{4y}=\frac{9}{5}$

II. In fractions of the type $\frac{a+b}{b}$ or $\frac{a-b}{b}$, performing componendo or dividendo,one can easily arrive at $\frac{a}{b}$

E.g. 15: If the sum of two numbers and the difference of the numbers are in the ratio 8 : 5, find the ratio of the numbers.

Noticing the sum and the difference, one should have thought of $(x + y)$ and $(x - y)$ immediately. And to find the ratio $x : y$ from ratio of above, one just needs to do componendo & dividendo. Thus the required answer is 13 : 3.

Law of Equal Ratios

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=......=k$, then by law of equal ratios, $\frac{a+c+e}{b+d+f}=k$

In fact not just this ratio, any identical 'linear' combination of numerators and denominators will also be k or in other words will be equal to each of $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ i.e. each of $\frac{a-c+e}{b-d+f},\frac{a-c-e}{b-d-f},\frac{2a-3c}{2b-3d},\frac{5c-a+2e}{5d-b+2f}$ or any such relation will also be equal to k or will be equal to each of $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

Logic & Extension

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots = k$. Thus, $a = bk$, $c = dk$, $e = fk$.

Consider $\frac{5c - a + 2e}{5d - b + 2f}$. It can be written as $\frac{5dk - bk + 2fk}{5d - b + 2f}$ i.e. $\frac{k(5d - b + 2f)}{5d - b + 2f}$ i.e. k.

In fact, while the above limits the application to any 'linear' combination i.e. no variable can be multiplied with each other nor can their power be taken, one can extend the logic to relations having powers of the variables as well ....

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots = k$, then by law of equal ratios, $\frac{a^n + c^n + e^n}{b^n + d^n + f^n} = k^n$. In other words,

$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots = \left(\frac{a^n + c^n + e^n}{b^n + d^n + f^n}\right)^{\frac{1}{n}}$. One could also multiply both numerator and denominator of each individual ratio by the same constant i.e. $\frac{k_1a^n + k_2c^n + k_3e^n}{k_1b^n + k_2d^n + k_3f^n} = k^n$

If one is adventurous, one can also frame relations of the type: If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$, then

$\frac{ac - e^2}{bd - f^2} = k^2$. One just needs to be careful that the degree of each term has to be the same and that the same relation of numerators has to be duplicated in the denominators. Anyways when in doubt, one can substitute $a = bk$, $c = dk$ and $e = fk$ and confirm the relation.

E.g. 16: If $\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$, find the ratio $\frac{a^2 - c^2}{b^2 - d^2}$. (Assume $a \neq c$)

As seen in above box, the required ratio will be $\left(\frac{4}{5}\right)^2 = \frac{16}{25}$.

For those who need to work it out, plugging $a = \frac{4}{5}b$ and $c = \frac{4}{5}d$, we have

$$\frac{a^2 - c^2}{b^2 - d^2} = \frac{\left(\frac{4}{5}b\right)^2 - \left(\frac{4}{5}d\right)^2}{b^2 - d^2} = \frac{\left(\frac{4}{5}\right)^2\left(b^2 - d^2\right)}{\left(b^2 - d^2\right)} = \left(\frac{4}{5}\right)^2$$

E.g. 17: If $\frac{a}{b} = \frac{c}{d} = \frac{4}{5}$, find the value of $\frac{4a - 3c}{5b + 4d}$.

Please note that in this case the relation of numerators, a & c i.e. $4a - 3c$ is not same as the relation of the denominators, b & d i.e. $5b + 4d$. So the above Law of Equal Ratios will not be applicable.

Assuming values for a, b, c and d (those satisfying the given condition) would be disastrous ...

If $a = c = 4$ and $b = d = 5$, then the required ratio will be $\frac{16-12}{25+20} = \frac{4}{45}$.

But with a different set of assumed values, $a = 4$, $b = 5$, $c = 8$ and $d = 10$,

the required ratio will be $\frac{16-24}{25+40} = -\frac{8}{65}$.

Thus, the required ratio will not have any unique value, its value will keep changing as different values are assumed for a, b, c, and d.

Theoretically, using $a = \frac{4}{5}b$ and $c = \frac{4}{5}d$, the required ratio will be

$\frac{4 \times \left(\frac{4}{5}b\right) - 3 \times \left(\frac{4}{5}d\right)}{5b + 4d} = \frac{16b - 12d}{25b + 20d}$, the value of which will depend on the relative values of b and d.

Exercise

21. What number must be subtracted from each of 7, 8, 11 and 14, so that the remainders are proportional.

    1. 1 2. 2 3. 3 4. 4 5. 5

22. What number must be added to each of 13, 25 and 45 such that the results are in continued proportion.

    1. 1 2. 2 3. 3 4. 4 5. 5

23. Find the mean proportional between 6 and 24.

    1. 9 2. 12 3. 15 4. 18 5. 21

24. If $\frac{a}{x-y} = \frac{b}{y-z} = \frac{c}{z-x}$, where none of the individual ratio is 0, find the value of $a + b + c$

    1. -1 2. 0 3. 1 4. $x + y + z$ 5. Not defined

25. Three natural numbers, a, b and c are in continued proportion. If $a + c = 5$, find the value of $(a^2 + b^2) : (b^2 + c^2)$.

    1. 1 : 4 2. 4 : 1 3. (1) or (2)

    4. More than 2 possible value 5. Conditions given not possible

Variation

Direct Variation

Speaking in general terms, two variables, x and y, are said to vary directly if when one of them increases, the other also increases **proportionally** and if one of them decreases, the other also decreases **proportionally**.

The word 'proportionally' is important here. If one of them doubles, the other should also double; if one of them becomes one-third, the other should also become one-third.

E.g. Speed and Distance covered in same time, say 1 hr, behave in the above manner. Consider I cover a certain distance at a certain speed. If the speed doubles, the distance covered also doubles and if the speed becomes one-third, the distance covered also becomes one-third.

Mathematically speaking...

Consider two variables x and y. As the variable x assumes different values, say x_1, x_2, x_3, ..., the variable y will also change and let's say it assumes y_1, y_2, y_3, ... values correspondingly.

If $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = ...$ then the two variables are said to vary directly. (This is a more precise definition than that stated while speaking in general terms. That one is more for a layman and is not absolutely accurate)

Conversely, if x & y vary directly, then $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \frac{x_3}{y_3} = ...$.

Another way of writing the above relation is that any pair of corresponding values (x, y) will satisfy $x = ky$, where k is a constant.

Direct relation is depicted as $x \alpha y$

In questions we would typically have only two instances of the variables, x and y, assuming different values e.g. in the question 'if 20 people can build 30 chairs, find the number of chairs that 50 people can build?', the variables are number of people and number of chairs and they vary directly. And there are two instances of this set of data – instance 1: 20 people, 30 chairs; instance 2: 50 people, x chairs.

One can use any of the following approaches:

Approach 1: Use ratios being equal i.e. $\frac{x_1}{y_1} = \frac{x_2}{y_2}$ or $\frac{x_1}{x_2} = \frac{y_1}{y_2}$ (both are the same, use any relation that one happens to write then).

Thus, in this case $\frac{20}{30} = \frac{50}{?} \Rightarrow$ $x = 75$. Or one can write the ratios as $\frac{20}{50} = \frac{30}{x}$, which again gives $x = 75$.

Approach 2: Use $x = ky$, where k is a constant.

One set of corresponding values (x, y) will be given – 20 people, 30 chairs. Use these values to find the value of the constant k.

$30 = 20k$ i.e. $k = 1.5$

Then use the relation again with, with the already found value of k, to find the unknown in the other set of data.

$x = 50 \times 1.5$ i.e. $x = 75$.

Approach 3 (Recommended):

Just find the multiplying factor in either of the pair: $x_1 \xrightarrow{\times ?} x_2$ or $x_1 \xrightarrow{\times ?} y_1$ and then use the same factor in a corresponding other pair.

You should verbalise this approach as follows for it to make more logical sense and not just maths:

20 people $\xrightarrow{\times 2.5}$ 50 people i.e. people are 2.5 times the earlier. So chairs will also be 2.5 times the earlier i.e. 30 chairs $\xrightarrow{\times 2.5}$ will be 75 chairs.

OR

20 people $\xrightarrow{\times 1.5}$ 30 chairs i.e. number of chairs are 1.5 times the number of people. Thus, 50 people $\xrightarrow{\times 1.5}$ will be 75 chairs.

While the recommended approach 3 will be used most often, in certain cases, we might have to use approach 2 as well, as shown by e.g. 21 below.

E.g. 18: y varies directly as x and when $x = 6$, $y = 24$. What is the value of y, when $x = 5$?

x changes from 6 to 5, will involve a fraction. Easier relation is that y (= 24) is 4 times x (= 6). Thus, when $x = 5$, then $y = 5 \times 4 = 20$.

E.g. 19: The value of a diamond varies directly as the square of its weight. If a diamond weighing 1.5 gms has a value of Rs. 9,000, what will be the value of a diamond weighing 2.5 gms?

Representing value by 'v' and weight by 'w', the relation is $v \propto w^2$. Thus one pair of given values is (9000, 2.25) and other pair if (v, 6.25). Since weight increases, $2.25 \xrightarrow{\times \frac{6.25}{2.25}} 6.25$, the value will also increase and the increase will be proportional. Thus, new value = $9000 \times \frac{6.25}{2.25} = 9000 \times \frac{25}{9} =$ 25000.

E.g. 20: The volume of a sphere is directly proportional to the cube of its radius. If the volumes of two spheres are in the ratio 8 : 1, find the ratio of the radii of the spheres.

Representing volume of the sphere by 'v' and radius by 'r', the ratio $\frac{v_1}{v_2}\left(=\frac{8}{1}\right)$ will be same as $\left(\frac{r_1}{r_2}\right)^3$. Thus, $\frac{r_1}{r_2}=\frac{2}{1}$

E.g. 21: The total surface area of a cylinder is the addition of the curved surface area and the area the base. The curved surface area is directly proportional to the radius of the cylinder and the area of the base is directly proportional to the square of the radius. If the total surface area of a cylinder is 72 sq. units when the radius is 4 units and is 105 sq. units when the radius is 5 units, find the radius of a cylinder having a total surface area of 144 sq. units.

Total Surface Area (T) = Curved Surface Area (C) + Base Area (B)

Since, $C = k_1 \times r$ and $B = k_2 \times r^2$, we have T = $k_1 r + k_2 r^2$

Putting the two pair of values of T and r, we get two equations:

$72 = 4k_1 + 16k_2$ ......(i)

$105 = 5k_1 + 25k_2$ .....(ii)

Solving (i) and (ii) simultaneously, we get $k_1 = 6$ and $k_2 = 3$.

Thus, plugging these values for the third pair of data we have $144 = 6r + 3r^2$ i.e. $r^2 + 2r - 48 = 0$ Solving the quadratic, $r = 6$ units.

Inverse Variation

Speaking in general terms, two variables, x and y, are said to vary inversely if when one of them increases, the other decreases **proportionally** and if one of them decreases, the other increases **proportionally**.

The word 'proportionally' is important here. If one of them doubles, the other should halve; if one of them becomes one-third, the other should become thrice.

E.g. Speed and Time taken to cover a certain distance behave in the above manner. Consider I take a certain time to go from home to office at a certain speed. If the speed doubles, the time taken will halve and if the speed becomes one-third, the time taken would become thrice.

Mathematically speaking ...

Consider two variables x and y. As the variable x assumes different values, say x_1, x_2, x_3, ..., the variable y will also change and let's say it assumes y_1, y_2, y_3, ... values correspondingly.

If $x_1 \times y_1 = x_2 \times y_2 = x_3 \times y_3 = \ldots\ldots$ then the two variables are said to vary inversely. (This is a more precise definition than that stated while speaking in general terms. That one is more for a layman)

Conversely, if x & y vary inversely, then $x_1 \times y_1 = x_2 \times y_2 = x_3 \times y_3 = \ldots\ldots$

Another way of writing the above relation is that any pair of corresponding values (x, y) will satisfy $x = \frac{k}{y}$, where k is a constant.

Inverse relation is depicted as $x \propto \frac{1}{y}$

In questions we would typically have only two instances of the variables, x and y, assuming different values e.g. in the question 'if a wall can be build in 10 days by 18 people, how many people will be needed to build the wall in 15 days?', the variables are number of people and days taken and they vary inversely. Further there are two instances of this set of data – instance 1: 18 people, 10 days; instance 2: x people, 15 days.

One can use any of the following approaches:

Approach 1: Use $x_1 \times y_1 = x_2 \times y_2$

Thus, in this case $18 \times 10 = 15 \times x$ i.e. $x = 12$.

Approach 2: Use $x = \frac{k}{y}$, where k is a constant.

One set of corresponding values (x, y) will be given – 18 people, 10 chairs. Use these values to find the value of the constant k.

$18 = \frac{k}{10}$ i.e. $k = 180$

Then use the relation again with, with the already found value of k, to find the unknown in the other set of data.

$x = \frac{180}{15}$ i.e. $x = 12$.

Approach 3 (Recommended):

Just find the multiplying factor among the two given values of the same variable: $x_1 \xrightarrow{\times ?} x_2$ or $y_1 \xrightarrow{\times ?} y_2$ and then use the reciprocal of the factor in two values of the other variable.

You should verbalise this approach as follows for it to make more logical sense and not just maths:

10 days $\xrightarrow{\times 1.5 \text{ or } \frac{3}{2}}$ 15 days i.e. days have become $\frac{3}{2}$ times the earlier. Thus, people will become $\frac{2}{3}$ times the earlier i.e. 18 people $\xrightarrow{\times \frac{2}{3}}$ 12 people.

While the recommended approach 3 will be used most often, in certain cases, we might have to use approach 2 as well, as shown by e.g. 25 below.

E.g. 22: If y varies inversely as x, when $x = 2$, $y = 3$. What is the value of x if $y = 1$?

Since y has become $1/3^{rd}$, x will become thrice i.e. $x = 6$.

E.g. 23: The number of four wheelers sold in a year vary inversely with the number of two wheelers sold in that year. If 4000 four wheelers were sold in one particular year, 21000 two wheelers were sold that year. How many two wheelers were sold in a year, when 6000 four wheelers were sold?

The two values are given for four wheelers. 4000 four-wheeler $\xrightarrow{\times \frac{3}{2}}$ 6000 four-wheelers i.e. the four-wheelers have become $\frac{3}{2}$ times the earlier. Hence two-wheelers have to become $\frac{2}{3}$ times the earlier i.e. 21000 $\xrightarrow{\times \frac{2}{3}}$ 14000.

E.g. 24: The cost per kg of rice varies inversely with the square of the quantity of rice produced in a year. When 7 million tons of rice was produced, its cost was Rs. 36/kg. How much was the production in the year when the cost of rice was Rs. 49/kg?

The two values are given for cost i.e. Rs. 36/kg has become Rs. 49/kg i.e. the cost has become $\frac{49}{36}$ times the earlier cost. Hence 'square of quantity', 49, will become $\frac{36}{49}$ times i.e. 'square of new quantity' is 36. Thus, 6 million tons is the required production.

E.g. 25: A designer gift store sells cube-shaped candles and the prices are quoted as per cubic inch of candle. Part of the cost per cubic inch of the candle varies inversely with the volume of the candle and rest of it varies inversely with the length of the side of the candle.The cost of a candle of side 2 inch is Rs. 16 per cubic inch. And the cost of a candle of side 3 inch is Rs. $5\frac{1}{9}$ per cubic inch. What is the cost per cubic inch of a candle of side 4 inches?

The cost per inch of the candle is made up of two parts, one which varies inversely with volume i.e. s^3 and other part that varies inversely with length of the side of the candle i.e. s.

Cost per inch = $\frac{k_1}{s^3} + \frac{k_2}{s}$

Putting the two pair of values of cost per inch and side, we get two equations:

$16 = \frac{k_1}{8} + \frac{k_2}{2}$ i.e. $k_1 + 4k_2 = 128$ and $\frac{46}{9} = \frac{k_1}{27} + \frac{k_2}{3}$ i.e. $138 = k_1 + 9k_2$.

Solving simultaneously, we get $k_1 = 120$ and $k_2 = 2$.

Now with $s = 4$, the cost per cubic inch = $\frac{120}{4^3} + \frac{2}{4} = \frac{15}{8} + \frac{1}{2} = \frac{19}{8}$ i.e.

Rs. 2.375 per cubic inch.

Joint Variation

There would be many situations in which more than two variables would be involved whereas the direct or inverse relation can be established only between pairs of variables (with others being constant). This section deals with how to link more than 2 variables in the same relation.

Consider three variables, x, y and z such that $x \,\alpha\, y$ and $x \,\alpha\, \frac{1}{z}$. The two individual relations $\frac{x}{y}$ = constant and $x \times z$ = constant can be combined into a same relation of $\frac{x \times z}{y}$ = constant.

Now two instances of data of x, y and z say (x_1, y_1, z_1) and (x_2, y_2, z_2) can be related as $\frac{x_1 \times z_1}{y_1} = \frac{x_2 \times z_2}{y_2}$.

Another example: If x varies inversely as y and y varies directly as square root of z, how would we link all three variables in the same relation?

The common variable between the two relation is y and thus, writing the individual relation in terms of y, we have $y \times x$ = constant and $\frac{y}{\sqrt{z}}$ = constant. Thus, the three are linked together as $\frac{y \times x}{\sqrt{z}}$ = constant.

E.g. 26: x varies directly with y as well as z. When $x = 2$ and $y = 3$, $z = 5$. What is the value of x when $y = 9$ and $z = 15$?

Method 1: $\frac{x}{y}$ = constant and $\frac{x}{z}$ = constant can be combined as

$\frac{x}{y \times z}$ = constant. Thus, $\frac{x_1}{y_1 \times z_1} = \frac{x_2}{y_2 \times z_2}$ and plugging in the values we

have $\frac{2}{3 \times 5} = \frac{x_2}{9 \times 15}$ i.e. $x_2 = 18$

Method 2: x varies directly with y. Since y has changed from 3 to 9 i.e. has become thrice, x will also become thrice i.e. x will change from 2 to 6.

Next since x varies directly with z and since z has also become thrice, 5 to 15, again x will become thrice i.e. x will be $6 \times 3 = 18$.

Direct & Inverse Relation

Direct Relation

Direct relation is distinct from Direct variation in the sense, an increase in one variable does cause an increase in second variable and a decrease in one variable also causes a decrease in other variable BUT the increase (or decrease) may not be Proportional.

Mathematically, if x and y vary directly, then $x = ky$, k is constant of proportionality. But if x and y are directly related, then $x = ky + k_1$, where k and k_1 are both constants.

To understand this, consider the problem: A class goes for a picnic to a resort. They hire a bus which costs Rs. 1000 for the entire trip. The resort charges Rs.100 per child for the lunch. If there are n children, the total cost (TC) for the trip would then be $1000 + 100n$.

TC = $1000 + 100n$. Here x is TC, k is 100 and k_1 is 1000.

So the total cost is directly related to the number of students. If $n = 20$, TC = 3000.

If the number of students double i.e. they become 40, TC = 5000. Note that now total cost has increased with the increase in the number of students but it hasn't doubled. This is because a part of the total cost remains fixed even if n increases.

So this example is of direct relation, not direct variation.

Let's take another example: Total manufacturing cost = cost per unit × number of units + fixed cost

TC = VC × n + FC, thus TC is directly related to n and the constants are VC (variable cost per unit, what I spend for each additional unit) and FC (fixed cost, irrespective of number of units).

Let's say because one extra unit is produced, the total cost increases by Rs.10. What inference can we gather from this? Why does the total cost increase? Remember that the Fixed cost remains constant.

It means the variable cost per unit, VC, is Rs.10.

Next, consider that the total cost increases by Rs.400 when 8 more units are produced. What changes happen since 8 more units are produced…the fixed cost remains same; it is just the additional cost per unit that is incurred for 8 units. Thus the variable cost per unit is Rs.400/8 = 50.

E.g. 27: Total boarding expenses of a boarding house are partly fixed and partly varying linearly with the number of boarders. The total boarding expenses is Rs. 1200 when there are 40 boarders and is Rs. 1500 when there are 55 boarders. What is the total boarding expenses when there are 75 boarders?

Total Cost(TC) = Fixed Cost(FC) + Variable Cost(VC) × Number of boarders(n)

1200 = FC + VC × 40…….(i)

1500 = FC + VC × 55……..(ii)

(i) and (ii) are simultaneous linear equations in 2 variables so you can solve them to get the value of FC and VC. We find that VC = 20 and FC = 400.

Thus, TC = 400 + 20 × n

When n = 75, the total cost = 400 + 20 × 75 = 1900

But read the box to understand the intellectually stimulating and faster method of doing it.

Logical way

The total boarding cost increased by Rs. 300 when 15 additional boarders joined in. This means that each boarder incurs a variable cost of Rs. 20 per boarders.

One does not even need to find the Fixed Cost. Total cost for 55 boarders is Rs. 1500. If 20 more join in, the cost will increase by 20 × 20 = 400. Thus total cost is Rs. 1900.

In case you need to find Fixed Cost: For 40 boarders, the total variable cost will be 40 × 20 = 800. But Total Cost is Rs. 1200. Thus Rs. 400 is Fixed Cost.

Now let's look at a variation of the above question:

E.g. 28: Total boarding expenses of a boarding house are partly fixed and partly varying linearly with the number of boarders. The average expense per boarder is Rs. 700 when there are 25 boarders and Rs. 600 when there are 50 boarders. What is the average expense per boarder when there are 100 boarders?

Will the above approach work?

With an increase of boarders from 25 to 50, the cost per boarder decreased from Rs. 700 to Rs. 600. i.e. additional 25 boarders decreases the cost per boarder by Rs. 100

Thus, when boarders increase from 50 to 100, an increase of 50, the cost per boarder will reduce by Rs. 200 i.e. it will reduce from Rs. 600 to Rs. 400.

Is this right? Without any answers, think what will be the cost per boarder if the number of boarders increases from 50 to 200 (or more) ...

Since boarders increases by 150, the cost per boarder should reduce by Rs. 600. Thus new cost per boarder will be Rs. 0 !!!

And if there are more than 200 boarders, the cost will decrease by more than 600 i.e. the cost per boarder will be negative – each boarder will receive money to put up there ☺

Obviously this approach is not suitable. Why?

The approach is suitable for a case of direct relation. This question is a case of inverse relation. With an increase in number of boarders (doubling), the cost per boarder decreases. Thus inverse proportion. But since it does not halve, the decrease is not proportion. This is a case of inverse relation.

We can solve this question in the way we solved the previous example as shown below:

When there are 25 boarders, total cost = 25 × 700 = 17,500

When there are 50 boarders, total cost = 50 × 600 = 30,000

Thus 25 additional boarders pay an extra total of Rs. 12,500 i.e. variable cost per person is Rs. 500.

For 50 additional boarders (from 50 to 100), total cost will increase by Rs. 50 × 500 = 25,000

Thus average cost per boarder when there are 100 boarders will be (30,000 + 25,000) / 100 = 550.

The following explains the logical way to solve the above without taking the route of total cost and working in inverse relation directly. However few students find the approach challenging and they can use the above solution also, it is also pretty simple.

Inverse Relation

TC = VC × n + FC

Dividing both sides by n, we get Cost per unit = VC + $\frac{\text{Fixed cost}}{n}$

This last quantity we call as the share of Fixed Cost that each unit bears.

Thus, Cost per boarder = $k_1 + \frac{k_2}{n}$

Thus, cost per unit is Inversely Related to number of unit. More the number of units, less will be the cost per unit. But the decrease will not be proportional to the increase because part of the cost per unit is constant.

Why will the cost per unit reduce? Each unit has a constant variable cost and a share of the total fixed cost. More units means less share of the fixed cost and hence cost per unit will reduce. Share of fixed cost is inversely proportional to number of units.

Continuing, we need to solve $700 = k_1 + \frac{k_2}{25}$ and $600 = k_1 + \frac{k_2}{50}$

Subtracting, $100 = k_2\left(\frac{1}{25} - \frac{1}{50}\right)$ i.e. k_2 = 5000. Plugging this back, k_1 = 500.

Thus, cost per boarder = 500 + $\frac{5000}{n}$.

With n = 100, cost per boarder = 500 + 50 = 550.

The above relation also shows that even if the number of students increases to a very very large number, the cost per boarder will never be lower than 500.

Solving logically

The decrease in cost per boarder, Rs. 700 – Rs. 600 = Rs. 100 is only because of each one's share of fixed cost.

And the share of fixed cost varies inversely to the number of boarders.

Since number of boarders are in ratio 1 : 2, the share of fixed cost is in ratio 2 : 1.

Hence we need two values in ratio 2 : 1 with a difference of Rs. 100. The values are Rs. 200 and Rs. 100.

Thus, when there are 50 boarders, the share of fixed cost is Rs. 100. When boarders double to 100, this share will reduce to 50 i.e. a decrease of 50.

Thus cost per boarder with 100 boarders is 50 less than 600 i.e. Rs. 550.

Though not needed, if we want to find the VC, with 50 boarders, share of fixed cost is Rs. 100 and total cost is Rs. 600. Thus, VC will be Rs. 500. Irrespective of the number of students coming, each has to pay Rs. 500. Thus, even if a very large number of students come, the cost per head will not reduce below Rs. 500.

Exercise

26. y varies directly as the square of x. When $x = 4$, $y = 2$. What is the value of y when $x = \frac{3}{2}$?

1. $\frac{9}{32}$ 2. $\frac{3}{4}$ 3. $\frac{16}{3}$ 4. $\frac{128}{9}$ 5. $\frac{7}{2}$

27. y varies inversely as the cube root of x. If $y = 7$, when $x = 8$, what is the value of x when $y = 56$?

1. $\frac{1}{2}$ 2. $\frac{1}{8}$ 3. $\frac{1}{3}$ 4. $\frac{1}{27}$ 5. $\frac{1}{64}$

28. y varies as the sum of two quantities, p and q. p varies directly as x and q varies inversely as x. If, $y = \frac{19}{3}$ when $x = 6$ and $y = \frac{33}{4}$ when $x = 8$, what is y when $x = 10$?

1. 57/5 2. 11 3. 51/5 4. 10 5. Cannot be determined

29. The value of a bar of gold weighing 400 gm, varies inversely with the square root of the fraction of impurities in it. If the value of a bar containing 25 gms of impurities is Rs. 45000, how many gms of pure gold is there in a bar costing Rs. 90000?

1. 12.5 2. 387.5 3. 375 4. 6.25 5. 393.75

30. The value of a silver coin varies directly as the square of its diameter, when thickness is constant and varies directly as its thickness when diameter remains constant. Two silver coins have the diameters in the ratio 4 : 3. Find the ratio of the thickness if the value of the first coin is four times the value of the second coin.

1. 3 : 4 2. 4 : 3 3. 16 : 3 4. 9 : 4 5. 1 : 3

31. The value of a diamond varies directly as the square of its weight. If a diamond falls and breaks into two pieces with weights in the ratio 2 : 3, what is the loss percentage in the value?

1. No loss 2. 16.66% 3. 25% 4. 40% 5. 48%

32. The product of the cost per head and the number of people going for a picnic is constant. When 10 more people join in the cost per head decreases from Rs. 40 to Rs. 36. What will be the decrease in cost per head if further 20 more people join in?

1. 2 2. 4 3. 6 4. 8 5. 12

33. x varies inversely as y and x varies directly as the square of z. If y decreases by 43.75% and z decreases by 75%, by what percent does x change?

1. 55.55% 2. 50% 3. 45.45% 4. 54.54% 5. 44.44%

Joint Variation (Continued) – Chain Rule

A certain type of problems involving joint variations are more commonly called problems on chain rule. They are an easier lot and also very easily identifiable as being questions on chain rule. Rather than use any theory to solve them, we will use pure logic to approach such problems and the logic is as explained in the following examples.

Let's start with a simple question.

If 10 weavers weave 10 carpets in 10 days, how many carpets will 1 weaver weave in 1 day?

If you have got your answer as 1 carpet, make sure you read the following very carefully, as it is not the correct answer.

10 weaver, 10 carpets, 10 days is one instance of the variables involved.

Had there been just 1 weaver i.e. the number of weavers reduce to $\frac{1}{10}$, obviously the number of carpets weaved would also reduce (proportionally)

Thus... 1 weaver weaves 1 carpet in 10 days.

If the number of days is reduced, from earlier 10 days to just 1 day now, obviously the 1 weaver would have weaved lesser carpet, to be exact $\frac{1}{10}$ of the carpet.

Thus, 1 weaver would weave $\frac{1}{10}^{th}$ carpet in 1 day.

Let's take another example:

E.g. 29: 4 carpenters make 20 chairs in 5 days. How many chairs will 8 carpenters make in 10 days?

4 carpenters make 20 chairs in 5 days. So 8 carpenters will make double the number of chairs in 5 days (same time as earlier) i.e. $20 \times \frac{8}{4} = 40$ chairs.

When you are accounting for the change in the number of chairs made because the number of carpenters has increased, you have to keep all the other factors (days available, etc) constant. Their effect will be included when that particular variable is taken as the focus.

Next since the number of days has also increased from 5 to 10 i.e .doubled, these 8 carpenters who earlier made 40 chairs in 5 days will make double the number of chairs in 10 days i.e. $40 \times \frac{10}{5} = 80$ chairs.

E.g. 30: 6 labourers can build a wall in 10 days. How many labourers of double the efficiency, are needed to build 3 such walls in 5 days?

To build 1 wall in 10 days, 6 laborers are required, so to build 3 walls in 10 days, 6 × 3 = 18 laborers will be required.

If the number of days in which the work has to be completed, decreases, the number of labourers required will increase. So when the days in which the work has to be completed reduce from 10 to 5, the men needed would increase proportionally i.e. 18 × 2 = 36 labourers would now be needed.

But since the new labourers have twice the efficiency, only half of them will be needed i.e. 18 laborers.

Taking an example, let's establish a method of doing these kind of questions:

E.g. 31: If 6 painters can paint 18 walls in 2 week, how many walls of twice the area can 16 painters paint in three weeks?

We are supposed to find the number of walls. Earlier 18 walls were painted. So start by writing '18 × ' on your sheet.

Now the number of painters has increased from 6 to 16. Thus more walls would be painted. Which of the ratio, $\frac{16}{6}$ or $\frac{6}{16}$, when multiplied with 18 will lead to more number of walls than 18? Obviously only when you multiply by a number greater than 1, will the number of walls increase, so it has to be $\frac{16}{6}$.

Hence since painters have increased from 6 to 16, the number of walls painted will be $18 \times \frac{16}{6}$. Don't calculate it as yet.

Next since the walls are twice the size, the number of walls painted will be less, to be precise will be $\frac{1}{2}$ of earlier i.e. $18 \times \frac{16}{6} \times \frac{1}{2}$.

Lastly since the number of weeks has increase from 2 to 3, one will be able to paint more walls. 'More' suggests a ratio greater than 1 i.e. $\frac{3}{2}$. Thus, the number of walls that can be painted is $18 \times \frac{16}{6} \times \frac{1}{2} \times \frac{3}{2}$ i.e. 36 walls.

E.g. 32: If 7 carpenters build 10 chairs in 12 days working 8 hours a day, how many days will it take for 4 carpenters to build 15 chairs if they work for 9 hours a day?

In the above question we need to find the number of days. So start with the number of days and then taking each of the other changing variable, one at a time, just spend a thought whether the number of days will increase or decrease because of the change. E.g. if number of carpenters decrease, number of days required to build the same number of chairs will be more. Also if number of chairs to be made increases, the number of days required to make them will also be more.

Accordingly multiply the number of days with the ratio of the variables that are changing i.e. number of carpenters and number of chairs here. Multiply with a ratio greater than 1 if the number of days has to increase and with the ratio less than 1 if the number of days has to decrease.

Days taken earlier

$$12 \times \frac{7}{4} \times \frac{15}{10} \times \frac{8}{9}$$

Hours working: increased from 8 to 9
More hours, so less days needed.
Thus ratio < 1 i.e. $\frac{8}{9}$

Carpenters: decreased from 7 to 4
Less men, so more days needed.
Thus ratio > 1 i.e. $\frac{7}{4}$

Chairs: increased from 10 to 15
More chairs, so more days needed.
Thus ratio > 1 i.e. $\frac{15}{10}$

Thus, number of days needed = $12 \times \frac{7}{4} \times \frac{15}{10} \times \frac{8}{9} = 28$.

Concept of man-days

Another way of approaching such problems is by quantifying the work as man-days or man-hours. E.g. If 7 carpenters build 10 chairs in 12 days working 8 hours a day, then the work i.e. 10 chairs can be quantified as involving 7 men × 12 days × 8 hrs/day i.e. 7 × 12 × 8 man-hours.

Needless to add, the work and required man-hours are directly proportional (if work doubles, the man-hours needed will double). Thus, if n, d, h and w refer to the number of men, days needed, hours per day and work respectively, then we can also use

$$\frac{n_1 \times d_1 \times h_1}{w_1} = \frac{n_2 \times d_2 \times h_2}{w_2}$$

This relation obtained is a result of joint variation. The individual relations $w \alpha n$; $w \alpha d$; $w \alpha h$ have all be combined into one joint variation i.e. $w \alpha n \times d \times h$ or

$$\frac{w}{n \times d \times h} = \text{constant.}$$

Use of this formulaic approach makes the solution less thought intensive, but then if other variables are introduced in the question, you will have to think where will the new variable appear - in the numerator or denominator.

E.g. 33: A garrison has enough food to sustain its 40 men for 6 months if each man consumes 900 gm of food per day. A reinforcement of 20 men join the garrison. For how long will the food last now, if each man now consumes only 600 gm of food per day?

Approach 1: Using proportionality directly

Days lasting earlier

$$6 \times \frac{4}{6} \times \frac{9}{6}$$

Men: increased from 40 to 60
More men, so food lasting less days
Thus ratio < 1 i.e. $\frac{40}{60}$

Intake: decreased from 900 to 600
Less intake, so food lasting more days
Thus ratio > 1 i.e. $\frac{900}{600}$

Thus the food will again last for 6 months.

Approach 2: Using man-days

One could also quantify the amount of food (equivalent of work in this example) in terms of man-days.

'enough food to sustain its 40 men for 6 months if each man consumes 900 gm of food per day' means that the food available is equivalent of 40 × 6 × 900 man-month-grams

Now when 20 more men join in, the number of men becomes 60 and since each eat 600 gm food each day, the available quantity of food will last for

$$\frac{40 \times 6 \times 900 \text{ man-month-grams}}{60 \times 600 \text{ man-grams}} = 6 \text{ months.}$$

Exercise

34. If 10 cows can eat 10 bags of grain in 10 days, how many days will it take for 1 cow to eat 1 bag of grain?

    1. $\frac{1}{100}$ 2. $\frac{1}{10}$ 3. 1 4. 10 5. 100

35. If 17 labourers can dig a ditch 20 m long in 18 days, working 8 hours a day, how many minimum number of more labourers should be engaged to dig a similar ditch 39 m long in maximum of 6 days, each labourer working 9 hours per day.

    1. 89 2. 84 3. 78 4. 72 5. 68

36. If 5 engines consume 6 metric tonnes of coal when each is running for 9 hours a day, how many metric tonnes of coal will be needed for 8 engines, each running 10 hours a day, given that 3 engines of the former type consume as much as 4 engines of the latter type.

    1. 8 2. 9 3. 12 4. 12.5 5. 14.2

37. The electricity bill of an office varies directly with the usage of the air conditioners. If 8 air conditioners of 1 ton each are used for 8 hrs every day, the monthly bill is Rs.5000. What will the approximate bill amount be, if 12 air conditioners of 1.5 tons each are used for 10 hrs every day?

    1. 8000 2. 9000 3. 10000 4. 12000 5. 14000

38. Four inlet pipes with circular cross section, take 9 hrs to fill a cistern. How many pipes of half the radius are required to fill the cistern in 6 hrs if the speed of water now is three times the speed of water in the previous case?

    1. 2 2. 4 3. 6 4. 8 5. 10

39. A garrison has sufficient food for 75 soldiers for a period of 90 days. After 10 days, one-third of the soldiers leave. After another 10 days, 5 soldiers return. From this day on, how many days will the food last?

    1. 80 2. 90 3. 100 4. 110 5. 120

40. A contractor undertakes to do a job in 20 days so he employs 36 laborers. After 5 days he realizes that three-eleventh of the work is over. To finish the work exactly on time, how many more laborers does he need to employ or how many can he afford to fire?

1. Hire 4 2. Hire 32 3. Fire 4 4. Hire 32 5. No change

41. A string 30 cm long when wound across a cylinder, makes $25\frac{1}{2}$ rounds. How many rounds would a string 45 cm long make when wound across a cylinder? The radii of the former and latter cylinder are in the ratio 4 : 3.

1. $28\frac{11}{16}$ 2. $12\frac{3}{4}$ 3. $22\frac{2}{3}$ 4. 51 5. None of these

42. Mike the mechanic has a machine which has four cog wheels in connection. The largest wheel has 242 teeth and the others have 66, 48 and 26, respectively. How many rotations must the largest wheel make before each of the wheels is back in its starting position?

1. 512 2. 312 3. 484 4. 660 5. None of these

CAT Questions

1. *[CAT 2006]* $\frac{a}{b} = \frac{1}{3}, \frac{b}{c} = 2, \frac{c}{d} = \frac{1}{2}, \frac{d}{e} = 3$ and $\frac{e}{f} = \frac{1}{4}$ then what is the value of $\frac{abc}{def}$

   (1) 3/8 (2) 27/8 (3) 3/4 (4) 27/4 (5) 1/4

Directions for 2 & 3: *[CAT 2006]* An airline has a certain free luggage allowance and charges for excess luggage at a fixed rate per kg. Two passengers, Raja and Praja have 60 kg of luggage between them, and are charged Rs. 1200 and Rs. 2400 respectively for excess luggage. Had the entire luggage belonged to one of them, the excess luggage charge would have been Rs. 5400.

2. What is the weight of Praja's luggage?

   (1) 20 kg (2) 25 kg (3) 30 kg (4) 35 kg (5) 40 kg

3. What is the free luggage allowance?

   (1) 10 kg (2) 15 kg (3) 20 kg (4) 25 kg (5) 30 kg

4. *[CAT 2004]* If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b} = r$ then r cannot take any value except:

   (1). 1/2 (2) –1 (3) 1/2 or –1 (4) –1/2 or –1

5. *[CAT 2003 – Retest]* Let a, b, c, d, and e be integers such that $a = 6b = 12c$, and $2b = 9d = 12e$. Then which of the following pairs contains a number that is not an integer?

   1. $\left(\frac{a}{27}, \frac{b}{e}\right)$ (2) $\left(\frac{a}{36}, \frac{c}{e}\right)$ (3) $\left(\frac{a}{12}, \frac{bd}{18}\right)$ (4) $\left(\frac{a}{6}, \frac{c}{d}\right)$

6. *[CAT 2002]* Mayank, Mirza, Little and Jaspal bought a motorbike for $60.00. Mayank paid one half of the sum of the amounts paid by the other boys. Mirza paid one third of the sum of the amounts paid by the other boys; and Little paid one fourth of the sum of the amounts paid by the other boys. How much did Jaspal have to pay?

   (1) 15 (2) 13 (3) 17 (4) None of these

7. *[CAT 2000]* A truck travelling at 70 kilometres per hour uses 30% more diesel to travel a certain distance than it does when it travels at the speed of 50 kilometres per hour. If the truck can travel 19.5 kilometres on a litre of diesel at 50 kilometres per hour, how far can the truck travel on 10 litres of diesel at a speed of 70 kilometres per hour?

   (1) 130 (2) 140 (3) 150 (4) 175

8. *[CAT 1999]* The speed of railway engine is 42 km per hour when no compartment is attached, and the reduction in speed is directly proportional to square root of the number of compartments attached. If the speed of the train carried by this engine is 24 km per hour when 9 compartments are attached, the maximum number of compartments that can be carried by the engine is

   (1) 49 (2) 48 (3) 46 (4) 47

9. *[CAT 1999]* Total expenses of boarding house are partly fixed and partly varying linearly with the number of boarders. The average expense per boarder is Rs. 700 when there are 25 boarders and Rs. 600 when there are 50 boarders. What is the average expense per boarder when there are 100 boarders?

   (1) Rs.550 (2) Rs.560 (3) Rs.540 (4) Rs.570

Percentages

Finding percent equivalent of '*a* out of *b*'

Percent literally means 'per cent', cent as in century. Thus, percent means 'per 100' and is represented with the symbol '%'.

This means that any data, *a* out of *b*, is proportionally reduced or increased so as to be 'out of 100'

E.g. A score of 80 marks out of 150 marks. To find the percentage of marks scored, we need to find the equivalent marks scored had the maximum marks been 100

80 marks out of 150

$\Rightarrow \frac{80}{150}$ marks out of 1 (Dividing by 150)

$\Rightarrow \frac{80}{150} \times 100$ marks out of 100. (Multiplying by 100)

Thus, 80 marks out of 150 is equivalent to a percent of $\frac{80}{150} \times 100$ i.e. 53.33%

E.g. In a class of 60 students, 45 of them are girls. To find the percent of girls in the class, we have to find the equivalent number of girls had the strength of the class been 100.

45 girls in a class of 60

$\Rightarrow \frac{45}{60}$ girls in a class of 1 (Dividing by 60)

$\Rightarrow \frac{45}{60} \times 100$ girls in a class of 100 (Multiplying by 100)

To convert the data '*a* out of *b*' to a percentage ......

...... we need to do the following: $\frac{a}{b} \times 100$

One needs to be slightly careful of what comes in the denominator. It is always the data associated with 'of' – 80 marks out **of 150** or 45 girls in a class **of 60**

E.g. 1: 50 is what percent of 80? Also find 80 as a percentage of 50.

As seen in the above examples, the required answer is $\frac{50}{80} \times 100 = \frac{250}{4}$ = 62.5%

While finding 80 as a percentage of 50, we would need to do: $\frac{80}{50} \times 100 =$ 160%

Finding x% of y

A percent figure is always a percentage **of** something. Thus, when we say the attendance was 80%, we mean that 80% of the total students were present; when we say that Rohit scored 85%, we mean 85% of the maximum marks; when we say that a number increased by 20%, we mean 20% of itself; when we say the profit was 40%, we mean 40% of the cost; when we say the sales grew by 35%, we mean 35% of the earlier sales figure.

To find x% of y ...

... we need to do $\frac{x}{100} \times y$

E.g. 2: In a poll, a total of 1,20,000 votes were polled of which 10% were invalid. A candidate received 60% of the valid votes. Find the number of votes polled by the candidate.

The number of invalid votes = 10% of 1,20,000 = $\frac{10}{100} \times 1,20,000$ = 12,000.

Thus, the number of valid votes = 1,20,000 – 12,000 = 1,08,000.

The candidate received 60% of 1,08,000 i.e. $\frac{60}{100} \times 1,08,000$ = 64,800.

E.g. 3: When 40% of a number is subtracted from 100, the answer is 80% of the number. Find the number.

If the number is n, then $100 - \frac{40}{100} \times n = \frac{80}{100} \times n$ i.e. $100 - 0.4n = 0.8n$ i.e. $1.2n = 100$ i.e. $n = 83.33$

E.g. 4: 100 is 30% of what number?

If the number is assumed to be n, $\frac{30}{100} \times n = 100 \Rightarrow n = \frac{1000}{3}$ = 333.33

Identifying the 'base' i.e. what the percentage is of

If is important to correctly identify the given percentage is of what number or the percentage asked is of what number. This number, of which the percent is found/given will be called as 'base'

Compare the following examples of similar situations, one straightforward and other very similar one but in which beginners commit errors. In the second example, the error is made in indentifying the base incorrectly.

...... cont'd

I.a. What is 20% of 450?

Here 20% of 450 is to be found, so 450 is your base of which 20% is to be found. Thus the answer is $\frac{20}{100} \times 450 = 90$

I.b. 20% of what number is 450?

Here you do not have to find 20% of 450. The number of which 20% is to be found is unknown. So if we assume the number to be n, the working would go as follows:

$\frac{20}{100} \times n = 450 \Rightarrow n = 2250$

In I.a the base is 450 and in I.b. the base is the unknown number to be found.

II.a. What is the result when 100 is increased by 20 %?

This should be a sitter. Here 100 is increased by 20 %, so we will find 20% of 100 and add it to 100. Thus the answer is 120.

II.b. A number increased by 20% becomes 100. Find the number.

The error is committed because of the presence of 100 and in haste most students find 20% of 100 as 20. And since the number has to be less than 100, the number is found as 100 – 20 = 80.

Please notice that 20% is of the number and not of 100. 100 is the final result after an increase of 100. Thus the correct approach should be assuming the number as n and then forming the equation $n + \frac{20}{100} \times n = 100 \Rightarrow \frac{6}{5}n = 100 \Rightarrow n = \frac{500}{6}$ = 83.333

In II.a the base is 100 and in II.b. the base is the unknown number to be found.

NOTE FOR NEXT EXAMPLE: In the case of profit percentage, the percentage is always of the cost price.

III.a. If a person purchases an article at Rs. 500 and sells it at a profit of 10 %, what is the selling price?

Cost Price = Rs. 500

Profit = 10% of CP = 10% of 500 = 50

Selling Price = 500 + 50 = 550

III.b. A person sells an article for Rs. 500 and makes a profit of 10%. Find his cost price.

Again in this case you CANNOT find 10% of 500 and subtract it from 500. Because his profit is 10% of the cost price and not of the selling price.

$\text{CP} + \frac{10}{100}\text{CP} = 500 \Rightarrow \frac{11}{10}\text{CP} = 500 \Rightarrow \text{CP} = \frac{5000}{11}$ = 454.5454

In III.a the base is 500 and in III.b. the base is the unknown CP to be found.

E.g. 5: Ram's salary is 90% of Shyam's salary. Shyam's salary is what percent of Ram's salary?

R is 90% **<u>of S</u>**. Thus, $\frac{R}{S} \times 100 = 90 \Rightarrow \frac{R}{S} = \frac{9}{10}$

The questions requires us to find S <u>as a percent of R</u> i.e. $\frac{S}{R} \times 100$. And the required answer will be $\frac{10}{9} \times 100 = \frac{1000}{9}$ = 111.11%

Alternately, Ram's salary is 90% that of Shyam's salary So assuming Shyam's salary as 100, we can get Ram's salary as 90% of 100 i.e. 90. Now the question is asking us to find 100 as a percent of 90 i.e. the base now is 90. Thus the answer is $\frac{100}{90} \times 100 = \frac{1000}{9} = 111.11\%$

E.g. 6: An alloy consists of 80% Bronze and rest Brass. Bronze and Brass, themselves are alloys with Bronze having 60% Copper and the rest Zinc whereas Brass has Copper and Zinc in equal proportions. What percentage of the alloy of Bronze and Brass is Zinc?

Let's assume the alloy of Bronze and Brass as 100 gms. Thus it will have 80 gms of Bronze and 20 gms of Brass. Let's find the amount of zinc in both. In 80 gms of Bronze, amount of Zinc is 40%. This 40% is of the amount of Bronze i.e. of 80 gms. Quantity of Zinc in Bronze = $\frac{40}{100} \times 80$ = 32 gms.

Similarly quantity of Zinc in Brass = $\frac{50}{100} \times 20$ = 10 gms.

Thus the total quantity of Zinc in the alloy is 32 + 10 = 42 gms. Now we have to find the percentage of Zinc in the alloy. The alloy is 100 gms and amount of Zinc in it is 42 gms. Thus the percentage of Zinc is 42%.

E.g. 7: Rohit scored 50% of the maximum marks in an exam and yet failed by 12 marks. Had he scored 10% more than what he scored, he would have just got passing marks. Find the maximum marks of the exam.

Let the maximum marks of the paper be x. Thus Raj scored $\frac{50}{100} \times x = \frac{x}{2}$

Now had he scored 10% more than what he scored means we have to calculate 10% of $\frac{x}{2}$ and this will be equal to 12 marks because he just managed to pass with these additional marks. Thus, we have $\frac{10}{100} \times \frac{x}{2} = 12 \Rightarrow x = 240$.

Alternately, if you wanted to avoid equations, 10% of the marks he scored is equal to 12. Thus he scored 120 marks. And this is 50% of the maximum marks. Thus, maximum marks = 240.

Exercise

1. When 30 is subtracted from a number, the result is 80 % of the number. Find the number.

   1. 150 2. 120 3. 100 4. 80 5. None of these

2. Mohan spends 40 % of his income on rent, 20 % on food, 10 % on entertainment and saves the rest. If he saves a net amount of Rs. 4380, find his income.

   1. 30,660 2. 21,900 3. 17,800 4. 15,200 5. 14,600

3. The prices of chocolates dropped by 15 % and thus Rashi was able to save Rs. 1.35 on each chocolate. Find the new price of one chocolate.

   1. Rs. 7.65 2. Rs. 8.65 3. Rs. 9 4. Rs. 9.35 5. Rs. 10.35

4. In a mixture of milk and water, milk accounts for 75 %. Find the amount of the mixture if it contains 28 lts of water.

   1. 21 lts 2. 56 lts 3. 84 lts 4. 112 lts 5. 168 lts

5. A milk-man mixes milk and water in the ratio 5 : 3. Find the percentage of milk in the resulting mixture.

   1. 60 % 2. 62.5 % 3. 67.5 % 4. 75 % 5. 80 %

6. When 1kg of goods is kept on an electronic weighing scale, the scale shows a reading of 950 gms. Find the percentage error in the measurement.

   1. 11.11 2. 5.55 % 3. 5 % 4. 5.26 % 5. 4.76 %

7. A unscrupulous trader has rigged his electronic weighing scale such that it shows a reading of 1 kg when 950 gms of goods are kept on it. Find the percentage error in the measurement.

   1. 11.11 2. 5.55 % 3. 5 % 4. 5.26 % 5. 4.76 %

8. A's salary is 80 % of B's salary. Find what percent of A's salary is B's salary.

   1. 120 % 2. 125 % 3. 140 % 4. 150 % 5. 180 %

9. To pass an exam of 250 marks, a candidate should obtain 40 % marks. Raj scores 10 % more than the passing marks. Find the percent of marks scored by Raj.

   1. 44 %   2. 50 %   3. 54 %   4. 60 %   5. 75 %

10. In a country 55.55 % of the population are males and rest females. Of the males 60 % are literate and of the females 40 % are literate. Find the percentage of the population that are literate.

    1. 50 %   2. 51.11 %   3. 55.55 %
    4. 45.45 %   5. Depend on the population

11. A milk man mixes water equal to 20 % of the milk he has. Water now accounts for what percentage of the mixture?

    1. 16.66 %   2. 20 %   3. 25 %   4. 22.22 %   5. 18.18 %

12. Mohan spends 40 % of his income on rent, 20 % on food, 10 % on entertainment and saves the rest. Find his savings as a percentage of his expenditure.

    1. 30 %   2. 33.33 %   3. 37.5 %   4. 40 %   5. 42.85 %

13. Anand has won 80 % of the games he has played so far in the tournament. His goal is to win 90 % of all the games he has to play in the tournament. If he has already played 15 out of the total 50 games that he has to play, what is the maximum number of games he can afford to loose in the remaining games and yet meet his goal?

    1. 0   2. 1   3. 2   4. 3   5. 4

14. *A*'s salary is 80 % of *B*'s salary whereas *B*'s salary is 80 % of *C*'s salary. What percentage of *A*'s salary is *C*'s salary?

    1. 64 %   2. 60 %   3. 140 %   4. 164 %   5. 156.25 %

Percentage-fraction equivalence

In the course of solving questions on percentages, we would very often have to find the percentages from expressions like $\frac{5}{6}\times 100, \frac{10}{11}\times 100, \frac{5}{8}\times 100$ or we would be finding 37.5% or 22.22% or 18.18% of a certain number. These could be tedious calculations and unnecessarily wear out a student. On the contrary these are very easy if one has already learnt a few percentage-fraction equivalence.

E.g. $\frac{2}{11}$ as a fraction is equivalent to 18.18%. This means that $\frac{2}{11}\times 100 = 18.18\%$ and 18.18% of n is same as $\frac{2}{11}\times n$

E.g. Knowing that $37.5\% \approx \frac{3}{8}^{th}$, the next time we have to find $\frac{3}{8}\times 100$, one can write the answer in a jiffy as 37.5%.

This will also help us in finding 37.5% of say 72. Rather than perform the tedious calculation $\frac{37.5}{100}\times 72$, one could just orally think of the expression as $\frac{3}{8}\times 72$ to arrive at 27 as the answer.

Fraction-Percentage

Throughout this book, quite often while finding a percent, the 100 will be ignored and use would be made only of the expression, $\frac{3}{8} = 37.5\%$.

Needless to say $\frac{3}{8}$ is actually equal to 0.375 and will be 37.5 only when expressed as a percentage i.e. $\frac{3}{8}\times 100 = 37.5\%$. But as stated, the 100 would be ignored quite often.

The following is a comprehensive list of the percentage-fraction equivalence that you would need to memorise. Groups that can be memorized together are given in one row.

$\frac{1}{2} = 50\%,\ \frac{1}{4} = 25\%,\ \frac{1}{8} = 12.5\%,\ \frac{1}{16} = 6.25\%$

$\frac{1}{3} = 33.33\%,\ \frac{2}{3} = 66.66\%$ $\frac{1}{6} = 16.66\%,\ \frac{5}{6} = 83.66\%$ $\frac{1}{12} = 8.33\%$

$\frac{1}{4} = 25\%,\ \frac{3}{4} = 75\%$

$\frac{1}{5} = 20\%,\ \frac{2}{5} = 40\%,\ \frac{3}{5} = 60\%,\ \frac{4}{5} = 80\%$

$\frac{1}{7} = 14.28\%,\ \frac{2}{7} = 28.56\%,$ $\frac{1}{14} = 7.14\%$

$\frac{1}{8} = 12.5\%,\ \frac{3}{8} = 37.5\%,\ \frac{5}{8} = 62.5\%,\ \frac{7}{8} = 87.5\%$

$\frac{1}{9} = 11.11\%,\ \frac{2}{9} = 22.22\%,\ \frac{4}{9} = 44.44\%,\ \ldots\ldots, \frac{7}{9} = 77.77\%$

$\frac{1}{11} = 9.0909\%,\ \frac{2}{11} = 18.1818\%,\ \frac{3}{11} = 27.2727\%,\ \ldots\ldots, \frac{7}{11} = 63.6363\%,\ \ldots\ldots,\ \frac{10}{11} = 90.9090\%$

$\frac{1}{15} = 6.66\%,\ \frac{1}{16} = 6.25\%$

$\frac{1}{19} = 5.76\%,\ \frac{1}{20} = 5\%,\ \frac{1}{21} = 4.76\%$

Practice the following to be swift in calculations (Answers are given on facing page at bottom)

1. Calculate the following percentages:

   a. 16.66% of 48 b. 12.5% of 180 c. 27.27% of 165

   d. 44.44% of 63 e. 6.25% of 24 f. 6.66% of 57

   g. 37.5% of 72 h. 83.33% of 42 i. 87.5% of 144

2. For each of the following find A as a percentage of B i.e. $\frac{A}{B} \times 100$

   a. A = 24, B = 288 b. A = 35, B = 56 c. A = 42, B = 66

   d. A = 63, B = 45 e. A = 20, B = 36 f. A = 88, B = 64

The Multiplying Factor

The concept of multiplying factor is the crux of this chapter. It will be useful to us in any scenario where percentages are involved – profit loss discount, simple & compound interest, growth rates in DI. So make sure you comprehend this very well.

Multiplying factor is a fraction, $\frac{a}{b}$, by which one will multiply the 'base' to find a percent of it or to find a number that is some percent more/less than it. We will denote the multiplying factor as *f* throughout this book, *f* to suggest it is a fraction.

Thus, the above states that to find a percent **<u>of *n*</u>** or a number that is some percent more or less **<u>than *n*</u>,** we would do the calculation $n \times f$

E.g. To find 9.0909% of *n*, we multiply *n* with $\frac{1}{11}$ i.e. do the calculation $\frac{1}{11} \times n$

To find 40% of *n*, we will do the multiplication $\frac{2}{5} \times n$

Percentage increase

Consider that a number *n* is increased by 22.22%. Find the expression for the value after the increase.

By now it would be evident that 22.22% is that of the number itself i.e. the increase is $\frac{2}{9} \times n$. This is just the increase, we have to find the final value after the increase.

Thus, this increase will be added back to the number i.e. the final value is $n + \frac{2}{9} \times n$

The process of finding 22.22% of *n* i.e. $\frac{2}{9}$ of *n* and then adding it back to *n* can be compressed into one step by considering our answer in the form $n \times \left(1 + \frac{2}{9}\right)$ i.e. $n \times \frac{11}{9}$

Thus increasing *n* by 22.22% is same as multiplying it by $\frac{11}{9}$. Thus we say that $\frac{11}{9}$ is the multiplying factor' corresponding to an 'increase of 22.22%'.

Answers to fraction percentage equivalence

1. a. 8	b. 22.5	c. 45	d. 28	e. 1.5	f. 3.8	g. 27	h. 35	i. 84
2. a. 8.33%	b. 62.5%	c. 63.63%	d. 140%	e. 55.55%	f. 137.5%			

Distinction between increase and final value

Say n is increased by 10% ……

10% of n or $\frac{1}{10} \times n$ is just the INCREASE in the value

The final value is n + 10% × n i.e. $\frac{11}{10} \times n$.

While the above is very easy, it is just a cautionary note asking you to notice carefully if one has to find '10% of n' or '10% more than n'

See if you understand finding the multiplying factors for a percentage increase as explained below:

Increase of 16.66% is equivalent to multiplying base by $\left(1+\frac{1}{6}\right)$ i.e. $\frac{7}{6}$

Increase of 37.5% is equivalent to multiplying base by $\left(1+\frac{3}{8}\right)$ i.e. $\frac{11}{8}$

Increase of 125% is equivalent to multiplying base by $\left(1+\frac{5}{4}\right)$ i.e. $\frac{9}{4}$

Percent Increase $\Rightarrow f$ is more than 1

In the case of a percent increase, the multiplying fraction, f will be more than 1. And this should obviously be so since only when a number is multiplied with a number greater than 1, would it increase. The 1 in the bracket $\left(1+\frac{a}{b}\right)$ refers to the original number and the $+\frac{a}{b}$ refers to the increase.

With the above logic …

$n \times f$, where f is greater than 1, will be a percentage increase. The amount by which the fraction is more than 1 will be the value of the percentage increase i.e. $(f - 1)$ will be equivalent to the percent increase, remember 1 refers to the original value and it should be subtracted from the final value to find the increase.

As explained in the above box, one should be able to work in the reverse way also.

$n \times \frac{8}{7}$ is same as n being increased by $\frac{8}{7} - 1 = \frac{1}{7}$ i.e. 14.28%

$n \times \frac{11}{9}$ is same as n being increased by $\frac{11}{9} - 1 = \frac{2}{9}$ i.e. 22.22%

Percentage Decrease

Consider we have to decrease n by 20%.

In this case the decrease will be $\frac{1}{5} \times n$ and since it is a decrease, it will have to be subtracted from n. Thus, final value = $n - \frac{1}{5} \times n$ i.e. $n \times \left(1 - \frac{1}{5}\right)$ i.e. $n \times \frac{4}{5}$

By the same logic ......

...... decrease of 11.11% is equivalent to multiplying base by $\left(1 - \frac{1}{9}\right)$ i.e. $\frac{8}{9}$

...... decrease of 27.27% is equivalent to multiplying base by $\left(1 - \frac{3}{11}\right)$ i.e. $\frac{8}{11}$

Percent Decrease $\Rightarrow$ f is less than 1

In the case of a percent decrease, the multiplying fraction f will be less than 1. And this should obviously be so since only when a number is multiplied with a number lesser than 1, would it decrease. The 1 in the bracket $\left(1 - \frac{a}{b}\right)$ refers to the original number and the $-\frac{a}{b}$ refers to the decrease.

$n \times f$, where f is less than 1, will be a percentage decrease. The amount by which the fraction is less than 1 will be the value of the percentage decrease i.e. $(1 - f)$ will be equivalent to the percent decrease, remember 1 refers to the original value and thus the final value has to be subtracted from it to find the decrease.

As explained in the above box, one should be able to work in the reverse way also.

$n \times \frac{7}{8}$ is same as n being decreased by $1 - \frac{7}{8} = \frac{1}{8}$ i.e. 12.5%

$n \times \frac{2}{5}$ is same as n being decreased by $1 - \frac{2}{5} = \frac{3}{5}$ i.e. 60%

Multiplying factors as decimals instead of fractions

In certain easy cases like 10 % increase or decrease, 20 % increase or decrease, one should also have the flexibility to use decimals instead of fractions i.e.

A 20 % increase should not only be thought as multiplying by $\frac{6}{5}$ but could also be thought as multiplying by 1.2, which is numerically exactly the same.

Similarly a 10 % decrease could also translate to a multiplying decimal of 0.9.

If multiplying by 0.9 or 1.2 is easier, use them else use the fractions. But be flexible enough and realize that they are the same.

Fill in the blanks in each of the following cases: (Answers at bottom of page)

1. n is increased by 8.33% $\Rightarrow n \times \frac{}{}$

2. Length of a rectangle is increased by 9.09% $\Rightarrow L_{new} = L_{old} \times \frac{}{}$

3. Prices of chocolates decrease by 12.5% $\Rightarrow P_{new} = P_{old} \times \frac{}{}$

4. Ram's marks are 30% less than Shyams' marks $\Rightarrow R = S \times \frac{}{}$

5. $a = b \times \frac{13}{11} \Rightarrow a$ is _____ % __________ than b

6. $\text{New Length} = \text{Old Length} \times \frac{15}{16} \Rightarrow$ Length has been ___________ by _____ %

Application of Multiplying Factor

The following text is a collection of solved examples involving standard questions about percentage changes especially when numeric values of underlying variables are not given.

The most typical example of such a question is "If x is 20% more than y, y is what percent less than x?". There are two equally easy ways to handle questions of this type as explained ......

Method 1: Using multiplying factors

In the data 'x is 20% <u>more than y</u>', the base is y and thus, $x = \frac{6}{5} \times y$

Whereas in the question phrase 'y is what percent <u>less than x</u>', the base is x. Thus, rearranging the above relation so that the base, x, is multiplied by a fraction, $y = \frac{5}{6} \times x$. This is equivalent to a $\left(1 - \frac{5}{6}\right)$ i.e. $\frac{1}{6}$ i.e. 16.66% decrease.

Method 2: Using numbers

Since the data given has the base as y, we can assume y = 100 and find x = 120.

Now the question requires us to find by what percent is 100 less than 120. And the answer will be $\frac{20}{120} \times 100$ i.e. $\frac{1}{6}$ i.e. 16.66%

Answers to fill in the blanks about multiplying fractions

1. 13/12 2. 12/11 3. 7/8 4. 7/10 5. 18.18%, more 6. decreased, 6.25%

E.g. 8: If A is 37.5% less than B, by what percent is B more than A?

Method 1: Using multiplying factors

$$A = \frac{5}{8} \times B \Rightarrow B = \frac{8}{5} \times A \text{ i.e. B is } \left(\frac{8}{5} - 1\right) \text{ i.e. } \frac{3}{5} \text{ i.e. 60\% more than A}$$

Method 2: Assuming numbers

Since we have to find 37.5% i.e. $\frac{3}{8}^{th}$ of a number, let B = 8. Thus A is 3 less than B i.e. A = 5.

We have to find 8 is what percent more than 5 i.e. $\frac{3}{5}$ i.e. 60%

Please note that in this example we did not assume B as 100 as that would lead to more calculations. Instead we used B = 8 since we had to find $\frac{3}{8}^{th}$ of B and this value of B would have made this calculation very very easy.

Formulaic Approach

While a formula for above types of questions also exists, we strongly recommend not to use the formula since it unnecessarily involves a bit of calculation

If a is x% more than b, then b is $\left(\frac{x}{100+x}\right) \times 100$% less than a

If a is x% less than b, then b is $\left(\frac{x}{100-x}\right) \times 100$% less than a

The same working as above would can also be used in standard questions of the type, "If the length of a rectangle is increased by 12.5%, by what percent should the breadth of the rectangle be decreased so that the area remains same" The solution is exactly similar to the working in above examples ...

Method 1: Using multiplying factors. If the original length and breadth is denoted as L and B respectively, equating the areas before increase and after increase,

$L \times B = \left(\frac{9}{8}\right) L \times (f) B$, we are using f as the multiplying factor for the change in breadth since we do not know the value of it.

From the above we get $f = \frac{8}{9}$ and this is equivalent to a decrease (less than 1) of $1 - \frac{8}{9}$ i.e. $\frac{1}{9}$ i.e. 11.11%

Method 2: Using numbers

If length was assumed as 100, the increased length would have been a not so friendly number, 112.5. Since length is increased by 12.5% i.e. by $\frac{1}{8}^{th}$, assume length as 8. Thus new length = 9.

To maintain the area, a smart work could be to assume the values of the original breadth as 9 and new one as 8 (i.e. original area = 8 × 9 and new area = 9 × 8) Thus, when breadth changes from 9 to 8, the percentage decrease is $\frac{1}{9}$ i.e. 11.11%

The area (or the product) need not always be constant as seen in the next example. Also the following example will depict that assuming numbers need not always be the best method and one should make an attempt to master the approach of using multiplying factors.

E.g. 9: The height and base of a triangle are changed such that even though the base is decreased by 10%, the area of the triangle increases by 10%. Find the percentage change in the height of the triangle.

Method 1: Using Multiplying factors.

Final Area = $\frac{11}{10}$ × Original Area

i.e. $\frac{1}{2}\times(f)\mathrm{H}\times\frac{9}{10}\mathrm{B}=\frac{11}{10}\times\left(\frac{1}{2}\times\mathrm{H}\times\mathrm{B}\right)\Rightarrow f=\frac{11}{9}$

Thus, the height is increased by $\left(\frac{11}{9}-1\right)$ i.e. $\frac{2}{9}$ i.e. 22.22%

Method 2: Assuming numbers

Since area increases by 10%, let's assume area as 10 and since base decreases by 10%, let's assume base = 10. Thus, $\frac{1}{2}\times 10\times h=10\Rightarrow h$ = 2, where h is the original height.

New area will be 11 and new base will be 9 and hence

$\frac{1}{2}\times 9\times h_1=11\Rightarrow h_1=\frac{22}{9}$, where h_1 is the new height.

Thus height increases from 2 to $\frac{22}{9}$ i.e. an increase percent of $\frac{\frac{22}{9}-2}{2}=\frac{2}{9}$

i.e. 22.22 %.

Relations of the type C = A × B

In the above question we have Area = Length × Breadth. There are many relations of this type and the context in questions like above could be any of these relations. The most commonly seen relation in questions on percentages is

Revenue (Or Expenditure) = Price × Quantity

Another very common application of above is in Data Interpretation, where

Sales = Market Share × Market

E.g. 10: If prices of sugar decrease by 6.25%, by what percent can a household increase its consumption of sugar in the same expenditure?

Method 1: Multiplying factor

If P and Q refer to the original price and quantity, we have

$$P \times Q = \left(\frac{15}{16}\right) P \times (f) Q \Rightarrow f = \frac{16}{15}$$

Thus the new quantity will be $\frac{16}{15}$ times the earlier quantity i.e. a percent increase of $\frac{1}{15}$ i.e. 6.66%

Method 2: Assuming numbers

Let price decrease from 16 to 15 (because it is a decrease of 1/16th and hence these values are chosen)

For product of price × quantity to remain same, we can smartly assume the quantity to increase from 15 to 16 i.e. an increase of 1/15 i.e. 6.66%

E.g. 11: When prices of milk went up by 18.18%, a household decreased the consumption of milk by 8.33%. By what percent did the expenditure of the family change?

As explained above, with such percentages, assuming numbers do not always result in easier calculation. Hence this question is being done only by multiplying factors.

If P and Q refer to the original price and quantity, then

$$\text{New expenditure} = \left(\frac{13}{11}\right) P \times \left(\frac{11}{12}\right) Q = \frac{13}{12} \times (P \times Q)$$

i.e. new expenditure is 13/12 times earlier expenditure. Thus, the expenditure increased by 1/12 i.e. 8.33%.

E.g. 12: Since prices of mangoes went up by 20%, a housewife was able to purchase 4 mangoes less in Rs. 60. Find the original price of each mango.

Though this question seems to be different from earlier, one should notice that the expenditure is the same, before and after increase.

Price have become 6/5 times of itself. Thus quantity will be come 5/6 times of earlier. This is $1/6^{th}$ less than earlier quantity and is given as 4 mangoes less. Hence originally the housewife was purchasing 24 mangoes in Rs. 60 i.e. a price of 60/24 i.e. 5/2 i.e. Rs. 2.5 for each.

If the question asked increased price, the housewife could now purchase 24 – 4 = 20 mangoes in Rs. 60 i.e. a price of Rs. 3 each.

If ratios is your strength area you could also think of the above as:

Since prices increased from, say 5 to 6, quantity can be assumed to be in the ratio 6 : 5. And we know that the difference in the quantity is 4 and hence the housewife was purchasing 24 earlier and now 20, both in Rs. 60. Thus original price of a mango will be 60/24 i.e. 5/2 i.e. Rs. 2.5.

Successive Percentage Changes

The population of a city was 80 lacs in year 2000. In the years 2001, 2002 and 2003 the population of the city increased by 20%, 16.66% and 10% respectively. What is the population of the city at the end of 2003.

In this question, there are three successive percentage increases. We say they are "successive" because for each percentage increase, the base is the value after the previous increase. In the year 2001, population grew to 80 × 1.2 = 96 lacs. Now in 2002, the 16.66% increase is over this new value 96 lacs and not over 80 lacs. Thus population at the end of 2002 is $96 \times \frac{7}{6}$ = 102 lacs and the population at end of 2003 is $102 \times \frac{8}{7}$ = 128 lacs.

All the above can be combined into the single expression: $80 \times \frac{6}{5} \times \frac{7}{6} \times \frac{8}{7}$ and without finding the populations in the intermediate years one could have found it directly for the last year as $80 \times \frac{8}{5}$ = 16 × 8 = 128.

E.g. 13: The population of a city increased by 25%, 18.18% 10% in three consecutive census but decreased by 9.09% in the fourth census. If the population after the fourth census was 65 lacs, find the population before the first census.

Here one would realise the importance of multiplying factors. Since the population of the base year is not given, it would be cumbersome to work unless one can use the following:

$$x \times \frac{5}{4} \times \frac{13}{11} \times \frac{11}{10} \times \frac{10}{11} = 65 \Rightarrow x \times \frac{65}{44} = 65$$ giving $x = 44$ lacs.

E.g. 14: If the population of a city changed by +12.5%, -12.5%, +6.66% over three consecutive years, find the total percentage change over the three years.

The net result of the three successive percentage changes is $\frac{9}{8} \times \frac{7}{8} \times \frac{16}{15} = \frac{21}{20}$ and $\frac{21}{20}$ is equivalent to a percentage increase of $\frac{1}{20}$ i.e. 5%.

Formula for two successive percent changes

The equivalent percentage change of two percentage changes a % and b % is $a + b + \frac{a \times b}{100}$ %.

While it is recommended to use the concept of multiplying factors, as explained in the examples, in certain cases where percent changes are very easy values like 10 % and 20 %, use of the formula can be resorted to.

The formula works well enough for percentage increase and also decrease if signs are taken correctly e.g. a 10 % increase and a 20 % increase will be a net percent of $10 + 20 + \frac{10 \times 20}{100}$ = 32 % whereas a 10 % increase and a 20 % decrease will be a net percent of $10 - 20 - \frac{10 \times 20}{100}$ = –12% i.e. 12% decrease.

Error-prone area

Two successive percent decrease of 10 % and 20 % cannot be thought of as $10 + 20 + \frac{10 \times 20}{100}$ = 32 % and then appending a minus sign because of the decreases i.e. a –32 %. This is wrong. The correct way is to assume any percent decrease as negative and work as $-10 - 20 + \frac{(-10) \times (-20)}{100}$ = –28 %. Notice how the third term becomes positive because of two negatives being multiplied together.

E.g. 15: What is the single percentage change equivalent to two percentage decreases of 10%.

Using multiplying factors, $\frac{9}{10} \times \frac{9}{10} = \frac{81}{100}$ i.e. a decrease of $\frac{19}{100}$ i.e. decrease of 19%

Using the formula, the equivalent percentage change is

$$-10 - 10 + \frac{(-10) \times (-10)}{100} = -19\%.$$

Exercise

15. If the length of a rectangle increases by 9.09%, by what percent should the breadth of the rectangle decrease to maintain the area of the rectangle?

    1. 8.33%   2. 9.09%   3. 10%

    4. 11.11%   5. Depends on actual length and breath

16. If the price of the entrance ticket of a circus is decreased by 6.25%, by what percent should the number of viewers increase so as to earn as much as that earned before the decrease in price?

    1. 5%   2. 5.555   3. 6.25%

    4. 6.66%   5. Depends on actual price

17. Traveling from home to office, if I increase my speed by 37.5% of my usual speed, what percent of time would I save?

    1. 22.22%   2. 27.27%   3. 33.33%   4. 37.5%   5. 45.45%

18. If both the length and the breadth of a rectangle increases by 15%, by what percent does the area of the rectangle increase?

    1. 15%   2. 30%   3. 32.5%   4. 22.5%   5. 37.5%

19. While preparing the price tag, instead of increasing the prices by 12.5%, a shopkeeper absent mindedly decreased the prices by 12.5%. Find the percentage difference in the price on the tag compared to what should have been the price tag.

    1. 25%   2. 22.22%   3. 1.5625%   4. 6.25%   5. 28.56%

20. Inspite of an increase in prices by 20%, Gopal managed to let his expenditures increase by just 10%. Find the percentage change in the quantity consumed.

    1. 10.2%   2. 10%   3. 9.09%   4. 9%   5. 8.33%

21. Since the prices of mangoes increased by 6.66%, the wholesaler was able to purchase 18 mangoes less in his budget of Rs 864. What is the increased price of each mangoes?

    1. Rs. 1.5   2. Rs. 2.5   3. Rs. 3   4. Rs. 3.2   Rs. 4

22. A large watermelon weighs 20 kg with 96% of its weight being water. It is allowed to stand in the sun for a while and some of the water evaporates so that now only 95% of it is water. Its reduced weight will be?

    1. 16 kg   2. 19.2 kg   3. 18 kg   4. 19.8 kg   5. 19 kg

Profit Loss Discount

The topic Profit, Loss and Discount is a straightforward application of Percentages. There is very little new matter that one would study.

Selling Price

In most of the problems there would be a trader who sells goods to customer. The price at which the trader sells the goods is called the Selling Price (SP, in short).

Cost Price

The trader would either be purchasing the good from else-where or manufacturing it. In either of the cases, he would be incurring a cost. This is called his Cost Price (CP, in short)

Profit/Loss

To find if the trade has resulted in a profit or loss, we would always consider the difference (SP – CP).

If the SP is greater than the CP, as usually is the case, the difference is positive and we say that the trader has earned a Profit.

If the SP is less than the CP, the difference is negative and we say the trader has incurred a Loss.

Profit or Loss = (SP – CP), if positive, it's Profit, if negative it's Loss

Profit/Loss Percentage

Whenever we have to compare two products, the absolute value of the profit is not of much use. A product may be resulting in a huge profit, say of Rs. 1000, as compared to another product which results in a profit of only Rs. 100. While the first product may seem attractive because of the higher absolute value of profit, but if the first product costs Rs. 10,000 and the second product costs only Rs. 200, the scenario changes. Having spent Rs. 10,000 and making a profit of only 1,000 ($1/10^{th}$ of the investment) is not as lucrative as spending only Rs. 200 and making a profit of Rs. 100 (1/2 of the investment). Thus, to compare two products, a better measure would be $\frac{\text{Profit}}{\text{Cost}}$. This figure is usually expressed in percentage terms i.e. profit per Rs. 100 of cost. This is termed as Profit Percentage. Thus, Profit Percentage = $\frac{\text{Profit}}{\text{Cost}} \times 100\,\%$

The same expression is used in the case of Loss Percentage as well. It's just that Profit in that case will be negative and the profit percentage will also be negative.

Profit is always expressed as a percentage of CP. Thus to find the Profit Percentage, the CP (and NOT the SP) will be the base.

E.g. 1: Find the profit percentage if the CP is Rs. 450 and the SP is Rs. 500

Obviously the profit = SP – CP = 500 – 450 = 50.

Profit Percentage = $\frac{50}{450} \times 100$ i.e. $\frac{1}{9} \times 100$ i.e. 11.11%

E.g. 2: A trader purchases a good at Rs. 500 and is forced to sell it at Rs. 450. Find his profit percentage.

In this case there is a loss i.e. Profit = 450 – 500 = –50

Profit percentage = $\frac{-50}{500} \times 100$ i.e. $-\frac{1}{10} \times 100$ i.e. 10% loss

E.g. 3: Amit sells his watch for Rs. 200 for a profit of 20%. At what price did Amit purchase the watch?

Please note that <u>20% is of the CP</u> which is not known and it is NOT of 200. Thus

CP + 20% of CP = 200

This is same as a 20% increase in CP results in 200. Thus using percentage increase fundas learnt in the earlier chapter, we have

$\frac{6}{5} \times \text{CP} = 200 \Rightarrow \text{CP} = \frac{1000}{6}$ = 166.66.

Profit Percentage is a percent increase

Throughout this text we would be using the funda of 'multiplying factor' for a percent increase or decrease. And we strongly recommend you also to do the same. Thus, rather than using Profit % = $\frac{\text{SP} - \text{CP}}{\text{CP}} \times 100$, we would use, CP × f = SP, where f is the multiplying factor related to the profit percentage.

Even when we have to find the profit percentage, we would make use of $f = \frac{\text{SP}}{\text{CP}}$ and then find the percentage increase/decrease corresponding to f. See the next example of why this is being done.

E.g. 4: Rohit purchases 10 articles for Rs. 3 and sells 15 articles for Rs. 4. Find his profit or loss percentage.

To find profit or loss percentage we need to know the CP and SP and thus finding them first,

CP = $\frac{3}{10}$ per article and SP = $\frac{4}{15}$ per article.

Now, working traditionally, to find the profit/loss percentage, we would have to do $\dfrac{\frac{4}{15}-\frac{3}{10}}{\frac{3}{10}} \times 100$ which is not very efficient.

As explained in the box above, we could just have found $f = \frac{SP}{CP} = \frac{4/15}{3/10} = \frac{8}{9}$.

And a multiplying factor of 8/9 means a percent decrease of 1/9 i.e. 11.11% loss.

We again reiterate that through-out this book ......

...... if profit/loss percentage is given we would use CP × f = SP

.... If profit/loss percentage is to be found, we will use $f = \frac{SP}{CP}$ and then find the equivalent percentage increase or decrease from f (except in the cases where the numbers used are very easy and oral calculation of profit percentage is possible)

E.g. 5: Ramesh sells his watch for Rs. 100 and makes a loss of Rs. 10. Find the loss percentage.

These easy questions are given just to rub in the fact that Profit or Loss percentage is always of CP and not SP. Thus in this questions 10 % loss is wrong answer as 10 is found as a percentage of 100, which is the SP and not CP.

CP = 100 + 10 = 110

Thus Loss percentage = $\frac{10}{110} \times 100$ i.e. $\frac{1}{11}$ i.e. 9.0909 %

Margin

Sometimes the profit is expressed as a percentage of SP. This percentage figure is called the Margin. Thus, Margin = $\frac{\text{Profit}}{\text{SP}} \times 100$

E.g. 6: If the profit percentage is 6.66 %, find the margin percentage.

6.66 % i.e. $\frac{1}{15}$ is of the CP, so it is best to assume the CP = 15. Hence profit = 1 and SP = 16. And now margin percentage is $\frac{1}{16}$ i.e. 6.25 %.

E.g. 7: If the margin percentage is –37.5 %, find the profit percentage.

In this question, –37.5 % is that of SP and NOT CP and further since it is negative it is a case of a loss. Since 37.5 % is $\frac{3}{8}$, let us assume the SP to be 8 and henc e loss = 3 and the CP = 11. Hence the profit percentage is $-\frac{3}{11}$ i.e. –27.27%.

Exercise

1. If the cost price is 25 % less than the selling price, find the profit percentage.

   1. 25 % 2. 27.27 % 3. 20 % 4. 32.5 % 5. 33.33 %

2. An article sold at a certain price results in a loss of 7 % whereas when it is sold at Rs. 100 more, it results in a 13 % profit. Find the cost price of the article.

   1. Rs. 400 2. Rs. 500 3. Rs. 600 4. Rs. 800 5. Cannot be determined

3. An article sold at Rs. 423 results in a loss of 10 %. What should be the selling price to result in a profit of 10 %?

   1. Rs. 443 2. Rs. 500 3. Rs. 507.6 4. Rs. 517 5. Rs. 550

4. A man gains 10 % when he sells the article at a certain price. Find his profit percentage if he increases the selling price by 50 %.

   1. 15 % 2. 60 % 3. 64 % 4. 65 % 5. Depends on the SP

5. Profit earned by selling an article at 1060 is 20 % more than the loss incurred by selling the article for Rs. 950. Find the cost price of the article.

   1. Rs. 975 2. Rs. 1000 3. Rs. 1020 4. Rs. 1040 5. Rs. 1050

Standard Problems on Profit and Loss

Chain of sale and purchase

E.g. 8: A manufacturer sells to the wholesaler at a profit of 20%. The wholesaler sells to the dealer at a profit of 12.5%. The dealer sells to a retailer at a profit of 10% and the retailer to the customer at a profit of 6.66%. If the cost of manufacturing the product for the manufacturer was Rs. 625, find the price paid by the customer.

All the profit percentages given are nothing by successive percentage increases on the cost of manufacturing. Thus the final price paid by the customer is $625 \times \frac{6}{5} \times \frac{9}{8} \times \frac{11}{10} \times \frac{16}{15} = 990$.

E.g. 9: A manufacturer sells to the wholesaler at a profit of 16.66%. The wholesaler sells to the dealer at a profit of 9.09%. The dealer sells to a retailer at a profit of 10% and the retailer to the customer at a profit of 14.28%. If the customer paid Rs. 468 for the article, find the cost of manufacturing the product incurred by the manufacturer.

In this question the final price is given and the base price, cost of manufacturing is not known. Thus if the cost of manufacturing is assumed to be x, we get

$$x \times \frac{7}{6} \times \frac{12}{11} \times \frac{13}{12} \times \frac{8}{7} = 468 \Rightarrow x \times \frac{13 \times 4}{3 \times 11} = 468 \text{ i.e. } x = 297.$$

Cost & Price given as Number of Articles/Rupee

E.g. 10: If the selling price of 10 articles is equal to the cost price of 9 articles, find the profit or loss percentage.

Method 1:

Let the SP of 10 articles = CP of 9 articles = Rs. k

SP of 1 article = Rs. $\frac{k}{10}$ and CP of 1 article = Rs. $\frac{k}{9}$

Using, $f = \frac{\text{SP}}{\text{CP}}$, we find the multiplying factor corresponding to the profit percentage as $\frac{9}{10}$ i.e. a loss of $\frac{1}{10}$ i.e. 10% loss.

Method 2:

To avoid all work involving fractions, a simple technique is to make use of LCM as follows:

Let the SP of 10 articles = CP of 9 articles = Rs. 10 × 9 say

SP of 1 article = Rs. 9 and CP of 1 article = Rs. 10

Thus profit percentage = $-\frac{1}{10}$ i.e. 10% loss.

E.g. 11: If the cost price of 16 articles is equal to the selling price of 15 articles, find the profit or loss percentage.

By the use of the LCM technique learnt above, CP of 1 article = Rs. 15 and SP of 1 article = Rs. 16. Thus profit percentage = 1/15 i.e. 6.66%

E.g. 12: By selling 12 articles, Naveen managed to make a profit equal to the selling price of 3 articles. Find the profit or loss percentage.

SP of 12 articles – CP of 12 articles = SP of 3 articles

SP of 9 articles = CP of 12 articles

Using the LCM approach, SP of 1 article = Rs. 12 and CP of 1 article = Rs. 9 and profit percentage is 3/9 i.e. 1/3 i.e. 33.33%

Alternately, $f = \frac{k/9}{k/12}$ i.e. $\frac{4}{3}$ i.e. a profit of 33.33%

E.g. 13: A person buys 12 oranges in a rupee. How many should he sell in a rupee so that he earns a profit of 20%.

CP of 1 orange = Rs. $\frac{1}{12}$

SP of 1 orange= Rs. $\frac{1}{12} \times \frac{6}{5} = \frac{1}{10}$

Thus 10 oranges have to be sold in 1 Re.

E.g. 14: A person buys 12 oranges for 5 Rs. How many oranges should he sell for Rs. 4 such that he makes a profit of 6.66%

CP of 1 orange = Rs. $\frac{5}{12}$

Profit of 6.66% will result in a multiplying factor of 16/15

Thus, SP of 1 orange = Rs. $\frac{5}{12} \times \frac{16}{15} = \frac{4}{9}$ and in Rs 4 he should sell $\frac{4}{4/9} = 9$

oranges.

E.g. 15: A person buys oranges at the rate of 5 for Rs. 2 and sells them at a rate of 7 for Rs. 3. Find the profit or loss percentage.

SP of 1 orange = Rs. $\frac{3}{7}$ and CP of 1 orange = Rs. $\frac{2}{5}$.

Using, $f = \frac{SP}{CP}$, we find the multiplying factor corresponding to the profit percentage as $\frac{15}{14}$ i.e. a profit of $\frac{1}{14}$ i.e. 7.14%.

Alternative Solutions

All the solutions to the above examples are based directly on theoretical approach. The given solution is not the only way to solve, one could also solve them innovatively, but the approach would have to be thought individually for each questions. Some alternate solutions for your developing your thoughts are given here ...

E.g. 10: SP of 10 articles = CP of 9 articles. On buying & selling 10 articles he recovers the cost of only 9 articles. Thus, he has made a loss equal to CP of 1 article when he bought and sold 10 articles. Thus, it is a case of loss and loss percentage = 1/10 = 10%

E.g. 11: CP of 16 articles = SP of 15 articles. On buying & selling 15 articles, he earned amount equal to CP of 16 articles. Thus, his profit is CP of 1 article. And profit percentage = 1/15 = 6.66%

E.g. 12: Selling 12 articles, the profit is equal to SP of 3 articles. Since the profit is given in terms of SP, we can find the margin percentage as 3/12 i.e. 25%. This is a profit percentage of $\frac{25}{100-25} = \frac{25}{75}$ = 33.33%

E.g. 13: He should earn Rs. 1.2 on selling the 12 oranges. Thus, in Rs. 1, he should sell 10 oranges (by proportionality)

E.g. 14: His cost is Rs. 5 and profit percentage is $\frac{1}{15}^{th}$. Thus, in Rs. $5\frac{1}{3}$ i.e. $\frac{16}{3}$ he sells 12 oranges. Again using proportionality, in Rs. 4 he should sell $12 \times \frac{3}{4}$ = 9 oranges.

E.g. 15: Buys in lots of 5 and sells in lots of 7 suggests that there would be no fractions if we assume quantity bought and sold to be 35. Thus, his cost will be Rs. 14 and earnings will be Rs. 15 i.e. a profit % of 1/14 i.e. 7.14%

Questions Involving Two or More Lots

E.g. 16: A trader sold one third of his stocks at a profit percentage of 10% and the rest at a profit percentage of 25%. Find his overall profit percentage.

Considering cost of his stock Rs. 300, he sold goods costing Rs. 100 at a profit of 10% and goods costing Rs. 200 at a profit of 25%. Thus his overall profit is Rs. 10 + Rs. 50 = Rs. 60.

Thus his overall profit percentage = 60/300 i.e. 1/5 i.e. 20%

Alternately, using funda of weighted averages (to be learnt in next chapter),

profit percentage = $\frac{10\% \times 1 + 25\% \times 2}{1+2} = \frac{60}{3}$ = 20%

E.g. 17: Ankur purchased oranges in two lots, one at rate of Rs. 3 per orange and other at rate of Rs. 5 per orange. He sold them at a rate of Rs. 4 per orange. Find his profit or loss percent if ......

Case i: he purchases the same number of oranges in the two lots.

Case ii: he spends the same amount of rupees on the two lots.

Case i: Since he purchases the same number of oranges in both the lots, let him purchase 1 orange in each lot. Thus his cost = Rs. 3 + Rs. 5 = Rs. 8.

He now sells 2 oranges at the rate of Rs. 4 per orange. Thus he earns Rs. 8

Thus he neither makes a profit nor a loss.

Case ii: In this case, he is spending equal amounts on each lot. Thus let him spend Rs. 15 (the LCM of 3 and 5) on each lot. Thus he has incurred of cost of Rs. 30 and purchased 5 + 3 = 8 oranges.

He sells these 8 oranges at a rate of Rs. 4 per orange and earns Rs. 32.

Thus his profit percentage = 2/30 i.e. 1/15 i.e. 6.66%

Profit percentage as weighted average of profit percentage on lots

Lot 1 is sold at profit of 33.33 % and lot two is sold at loss of 20 %. So should the overall profit percentage not be an average or them? Rather should it not be the arithmetic mean of them, $\frac{33.33+(-20)}{2} = \frac{13.33}{2}$ = 6.66%?

Read the following if you are aware of concept of weighted average. Or come back to this once chapter on weighted averages is done.

When there are two or more lots, sold at different profit percentages, the overall profit percentage will obviously be the average of the profit percentages that the individual lots are sold at. But it is going to be a case of weighted average and not arithmetic mean. And the weights are going to be costs incurred in the two lots and not number of items in the two lots.

Case ii: In this case, since the cost incurred in the two lots are equal, i.e. ratio of weights is 1 : 1, the overall profit percentage will be $\frac{33.33\times1+(-20)\times1}{1+1} = \frac{33.33+(-20)}{2}$ i.e. the arithmetic mean only because the weights were equal.

Case i: Though the ratio of the number of oranges bought in each lot is 1 : 1, the weights will not be equal and the overall profit percentage will not be the arithmetic mean. Since quantity is in the ratio 1 : 1 and cost price is in ratio 3 : 5, the costs incurred on the two lots will be in the ratio 3 : 5 and thus overall profit percentage = $\frac{33.33\times3+(-20)\times5}{3+5} = \frac{100+(-100)}{8}$ = 0 %.

E.g. 18: Ankur purchased oranges in two lots, one at rate of 3 oranges for a rupee and other at rate of 5 oranges for a Rupee. He sold them at a rate of 4 oranges per Rupee. Find his profit or loss percent if ......

Case i: he purchases the same number of oranges in the two lots.

Case ii: he spends the same amount of rupees on the two lots.

This question differs from the earlier example because here the rate is given as number of oranges per rupee and not rupees per orange.

Case i: Same number of oranges are purchased in each lot. First lot is of 3 oranges per rupee and second is of 5 oranges per rupee. Also the selling rate is of 4 oranges per rupee. Thus let him purchase the LCM of 3, 4 and 5 oranges in each lot i.e. 60 oranges in each lot. This would avoid all fractions.

So he purchases 120 oranges in all and incurs a cost of Rs. 20 + Rs. 12 = Rs. 32.

He sells these 120 oranges, 4 to a rupee and thus earns Rs. 30. Thus it is a case of loss.

Loss percentage = 2/32 i.e. 1/16 i.e. 6.25% loss.

Case ii: Let him spend 1 rupee on each lot. Thus he purchases 3 + 5 = 8 oranges after spending Rs. 2.

He sells the 8 oranges, 4 to a rupee and thus earns Rs. 2

So, it’s a case of him not earning any profit nor losing any money.

E.g. 19: Rohan buys two commodities for Rs.60000 each. He sells one at a profit of 20 % and sells the other at a loss of 20 %. Find his overall profit or loss percent and also the amount of profit or loss.

Since the CP of both the articles are the same, +20% of 60,000 and –20% of 60,000 will cancel out each other and he will neither make any profit nor any loss.

E.g. 20: Rohan sells two commodities for Rs. 19,800 each. He sells one at a profit of 10 % and sells the other at a loss of 10 %. Find his overall profit or loss percent and also the amount of profit or loss.

Two items, one at profit of x % and other at loss of x %

In case like this when the SP of two articles are the same and one is sold at a profit of x % and other at a loss of x %, there would always be a loss of $\frac{x^2}{100}$ %

However make sure that the selling price is same and that that profit percentage and loss percentage are numerically equivalent. Compare the question with e.g. 19, when the cost price was same. In this case the overall there is no profit no loss. Compare it with the approach used in next e.g. 21, when the profit and loss percent are different.

Thus loss percent in this case = $\frac{10^2}{100}$ = 1% loss.

To calculate the amount of loss, we would have to find the CP of the articles. Thus,

$CP_1 \times \frac{11}{10} = 19,800 \Rightarrow CP_1$ = 18,000 and $CP_2 \times \frac{9}{10} = 19,800 \Rightarrow$

CP_2 = 22,000

Thus total cost = Rs. 18,000 + Rs. 22,000 = Rs. 40,000

Total earnings = Rs. 19,800 + Rs. 19,800 = Rs 39,600.

Thus loss = Rs. 400.

Alternately after finding the costs, loss = 1% of total cost = 1% of 40,000 = Rs. 4,000

E.g. 21: Rohan sells two commodities for Rs. 12,000 each. He sells one for a profit of 20% and other at a loss of 11.11%. Find his overall profit or loss percentage and also the amount of profit or loss.

While the selling price of the two commodities in this question is the same, but the profit % and loss % on the two articles is not the same numeric percentage and hence we cannot use the short-cut learnt in earlier question. In this case, we would have to work out the costs and only then find the profit/loss percentage.

$CP_1 \times \frac{6}{5} = 12,000 \Rightarrow CP_1$ = 10,000 and $CP_2 \times \frac{8}{9} = 12,000 \Rightarrow CP_2$ = 13,500

Thus total cost = Rs. 23,500 and total earnings = Rs. 24,000.

Thus profit percentage = 500/23,500 i.e. 1/47 i.e. $\frac{1}{47} \times 100$ i.e.

approximately 2.1%

With Changes in the CP and SP

E.g. 22: When the cost price of an article increases by 20%, a shop-keeper also increases his selling price by 20%. Find the change in the profit percentage and also in the amount of profit.

The profit percentage is solely dependent on the ratio $\frac{SP}{CP}$. With a 20% increase in both CP and SP, the ratio will not change and hence the profit percentage will also not change.

Hence we see that if both cost price and selling price are increased by the same percentage, the profit percentage remains the same. Since profit percentage is the same, but cost price has increased, so the amount of profit would also have increased. Let's see by how much.

Earlier Profit = SP – CP

New Profit = 1.2SP – 1.2CP = 1.2 × (SP – CP) i.e. 1.2 times the earlier profit.

Hence the profit amount increases by 20% when both SP and CP are increased by 20%

E.g. 23: When the cost of an article increases by Rs. 170, a trader increases his selling price by 10%. Because of these changes his profit percentage decreases from 20% to 15%. Find the cost price of the article after the increase.

Let the original cost price of the article by CP. Since his original profit percentage was 20%, the original selling price must have been 1.2 × CP.

Because of the changes, his new cost would be CP + 170 and the new selling price would be (1.2 × CP) × 1.1. Since the new cost and selling price yield a profit of 15%, we have $\frac{(1.2 \times CP) \times 1.1}{CP + 170} = 1.15$

i.e. 1.32 × CP = 1.15 × CP + 1.15 × 170

i.e. 0.17 × CP = 1.15 × 170 i.e. CP = 1150

Exercise

6. When the cost of an article increases by Rs. 340, a trader increases his selling price by 10%. Because of these changes his profit percentage decreases from 20% to 15%. Find the cost price of the article after the increase.

   1. Rs 3600 2. Rs 2300 3. Rs 3940 4. Rs 3260 5. Rs 2640

7. On selling 20 mts of cloth, a merchant realizes that he has made a loss equal to the selling price of 3 mts of cloth. Find his loss in percentage terms.

   1. 15% 2. 13% 3. 16.66% 4. 20% 5. 23%

8. The cost price and selling price is such that the cost price of 36 articles is equal to the selling price of x articles. Find the value of x that will result in a profit of 12.5%

   1. 40 2. 36 3. 32 4. 27 5. 24

9. A trader sells 10 articles for a rupee and manages a profit of 20%. How many articles did he purchase in a Rupee?

   1. 9 2. 10 3. 11 4. 12 5. 15

10. A trader purchases orange at the rate of 1 dozen for Rs. 5. How many oranges should he sell per Rupee such that he makes a profit of 20 %?

    1. 2 2. 3 3. 4 4. 5 5. 6

11. A trader buys 20 kgs of wheat at the rate of Rs. 6 per kg. He sells the wheat at the rate of 8 per kg. But, for a regular customer his rate is Rs. 5 per kg. On selling the entire 20 kgs, he realizes he has not made any profit or loss. How many kgs were bought by regular customers?

    1. 20/3 kgs 2. 20/6 kgs 3. 40/3 kgs 4. 40/6 kgs 5. None of these

12. A trader buys articles at the rate of 10 per rupee. He sells one third of the lot at the rate of 12 per rupee and the rest at the rate of 9 per rupee. Find his profit percentage.

    1. 10% loss 2. 20% profit 3. 10% profit 4. 6.66% profit 5. No profit no loss

13. Articles were bought at the rate of 6 for Rs. 5 and sold at the rate of 5 for Rs. 6. Find the profit percentage.

    1. 44% 2. 44.44% 3. 45.45% 4. 40% 5. 41.111%

14. A shop-keeper sells two articles, each for Rs. 1958. If he sold one at a profit of 10% and the other at a loss of 10%, find the amount of profit or loss.

    1. Rs. 0 2. Rs 19.58 loss 3. Rs 39.16 loss 4. Rs 39.5 loss 5. Rs 40 loss

15. A man sells an article at a profit of 20%. If he bought it at 10% less and sold it at Rs. 18 more he would make a 40% profit. Find the cost price of the article?

    1. Rs. 100 2. Rs. 200 3. Rs. 300 4. Rs. 400 5. Rs. 500

Mark-Up and Discount

Since customers haggle for a discount, the usual practice is that shop-keeper's prepare the price tag that is higher than the acceptable price at which the shop-keeper is willing to sell at. Then a discount is offered on the Marked Price (MP), also called List Price and the article is sold at a Selling Price lower than the Marked Price. The profit is still calculated based on the final Selling Price and the Cost Price. The associated terms are defined below:

Marked Price:

This is the price that is marked on the price tag

Markup:

The percentage by which the MP is higher **than the CP.** Thus mark-up percentage is a percentage of the Cost Price.

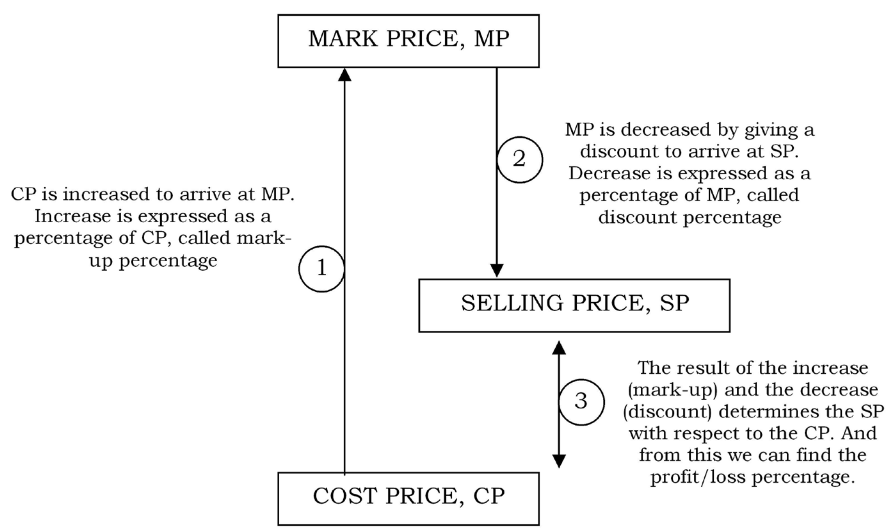
Mark-up should be viewed as just a percentage increase is. Thus, for a particular mark-up percentage, there would be an associated multiplying factor, say f_m and we would have the relation CP × f_m = MP

Discount:

Discount is the amount by which the MP is lowered while selling due to sales promotion or due to bargaining. It is usually expressed as a percentage and is a percentage of the Marked Price.

Discount should be viewed as just a percentage decrease is. Thus, for a particular discount percentage, there would be an associated multiplying factor, say f_d and we would have the relation MP ×f_d = SP

The diagram pictorially represents the above terms:



Mark-up & Discount as case of Successive Percent Changes

Let's denote mark-up percentage by m % and discount percentage by d %.

It should be apparent that

m % is a percentage increase over the CP to result in the MP

d % is a percentage decrease over MP to result in SP.

Thus, first CP is increased by m% to result in the MP and then the MP is decreased by d % to result in the SP i.e. m % and d % are two successive percentage changes acting on CP to result in SP

Thus using the formula learnt for successive percentage changes in the chapter on percentages we can say that:

Profit percentage = $m - d - \frac{m \times d}{100}$ %

As stated in chapter of percentages, the above is useful only when mark-up and discount are good numbers like 10% or 20%. If they are not so good numbers like 12.5% or 16.66%, using multiplying factor is strongly recommended. If the multiplying factors related to profit, mark-up and discount are denoted as f_p, f_m, f_d respectively,

MP = CP × f_m and SP = MP × f_d

Thus, SP = CP × f_m × f_d and we already know that SP = CP × f_p

Thus, $f_p = f_m \times f_d$

E.g. 24: A trader marks-up his goods by 30% and then offers a discount of 10%. Find the net profit or loss percentage that he makes.

Profit percentage = $30 - 10 - \frac{30 \times 10}{100}$ = 17%

E.g. 25: A trader marks-up his goods by 27.27% and then offers a discount of 8.33%. Find his profit percentage.

Since the numbers are not very amenable in the formula, working on multiplying factors,

$f_p = \frac{14}{11} \times \frac{11}{12} = \frac{7}{6}$. Thus, the profit percentage is 1/6 i.e. 16.66%

The above should be self-sufficient, but if you are not convinced about $f_p = f_m \times f_d$, see if the following helps.

$$\underbrace{\left(\mathrm{CP} \times \underbrace{\frac{14}{11}}_{\substack{\text{marking-up}\\ \text{\% inc of 3/11}}}\right)}_{\text{Mark Price}} \times \underbrace{\frac{11}{12}}_{\substack{\text{discounting}\\ \text{\% dec of 1/12}}} = \mathrm{SP}$$

i.e. $\mathrm{CP} \times \frac{7}{6} = \mathrm{SP}$ which suggests a profit percentage of 1/6 i.e. 16.66%

E.g. 26: Inspite of giving a discount of 9.09%, a trader made a profit of 11.11%. By what percentage did he mark-up his goods?

Using $f_p = f_m \times f_d$, we have $\underbrace{\frac{10}{9}}_{\text{profit } 1/9} = f_m \times \underbrace{\frac{10}{11}}_{\text{discount } 1/11} \Rightarrow f_m = \frac{11}{9}$

A multiplying factor of 11/9 means a percentage increase of 2/9 i.e. 22.22%

Exercise

16. A 25% discount offer results into a saving of Rs. 37. Find the selling price of the article.

    1. Rs 99 2. Rs 100 3. Rs 101 4. Rs 111 5. Rs 74

17. A trader gives two successive discounts of 20% and 10%. What is the equivalent discount that he is offering?

    1. 30% 2. 33% 3. 25% 4. 21% 5. 28%

18. A scheme of 1 soap free with every 4 soaps purchased is launched for increasing the sales. What is the effective discount that the scheme offers?

    1. 33.33% 2. 37.5% 3. 25% 4. 20% 5. None of these

19. As a sales incentive, which of the following two schemes should a shampoo manufacturer prefer over the other?

    I: Offer to give 25% more quantity for the same price; II: A discount of 25% on the price.

    1. I 2. II 3. Both are same

20. What should be the mark-up percentage if a trader wishes to make a profit of 10% inspite of a discount of 10%

    1. 20% 2. 18.18% 3. 22.22% 4. 27.27% 5. 25%

21. Find the ratio of the marked prices of two articles whose selling prices are same after they are sold at a discount of 12.5% and 9.09% respectively.

    1. 11 : 8 2. 33 : 32 3. 80 : 77 4. 41 : 40 5. 27 : 22

22. The ratio of marked price and the cost price of an article are in the ratio 3 : 2. If $2x$% discount is given on the article then x% loss is incurred. What is the value of x%?

    1. 10% 2. 16% 3. 20% 4. 25% 5. Cannot be determined

23. After allowing a discount of 25% on the marked price, a shopkeeper charges Rs 450 for a watch. Had he not allowed any discount he would have made a profit of 20%. What was the cost price of the watch?

    1. Rs. 100 2. Rs. 200 3. Rs. 300 4. Rs. 400 5. Rs. 500

Rigged Balances and Faulty Weights

Unscrupulous traders, in addition to selling at a price higher than his cost price (which is not why he is unscrupulous), also cheats on the volume by using a rigged scale or weight. Thus increasing his profits further. This section here deals with finding the profit percentage because of rigged balances or faulty weights.

E.g. 27: A trader sells at the same rate at which he purchase his goods. However he uses a balance which reads 1000 gms for 900 gms. Find his profit percentage.

The crucial statement to understand is 'reads 1000 gms for 900 gms'. This means that when goods weighing 900 gms are kept, the balance will show a reading of 1000 gms. Thus the customer will pay for 1000 gms, as the balance is showing that, but will receive only 900 gms. Thus the cost to the trader is for 900 gms only. Let's say the rate at which the trader purchases and sells is Rs. x per kg.

Thus earnings of the trader = 1000 gms × x = $1000x$ and

cost to the trader = 900 gms × x = $900x$.

Thus profit percentage = 100/900 i.e. 1/9 i.e. 11.11%

In case of Faulty Balances ......

...... keeping in mind the following will help you arrive at the solution very easily:

1. Keep in mind whose perspective the question requires you to work with.

With a shop-keeper's perspective ......

...... the cost will be associated with the 'actual' weight of goods given.

...... the earnings will be associated with the 'reading' on the balance, because a customer will pay for as much as the balance reads.

Whereas with a customer's perspective ......

...... the cost to her is the actual amount paid which will depend on the 'reading' on the balance

...... and the goods she received, sort of her earnings, will be 'actual' weight.

2. Never assume that the shopkeeper will always be at an advantage i.e. earning profit due to faulty weights. It might also be the case that the shopkeeper is unaware that his balance is faulty and hence could also be making a loss. So use point 1 to determine how to find the cost and earning, rather than thinking the higher value will be for earnings and lower value for costs.

The best way is to consider that the shopkeeper is also unaware of the fact that the balance does not read correctly. Don't consider it to be his deliberate action.

E.g. 28: A trader sells at a mark-up of 10% and at the same time uses a 1 kg that weighs only 900 gms. Find his profit percentage.

This question is also similar to the earlier question in the sense that the customer pays for 1 kg but receives only 900 gms of goods. Additionally there is also a mark-up that the trader applies i.e. his rate of selling is 10% more than his rate of purchase. Thus if he buys goods at the rate of Rs. x per gm, he sells goods at the rate of Rs. $1.1 \times x$ per gm.Thus,

earnings of the trader = 1000 gms × $1.1x = 1100x$, and

cost of the trader = 900 gms × $x = 900x$.

Thus profit percentage = 200/900 i.e. 2/9 i.e. 22.22%

Varieties of Faulty Weights

Approach of solving questions where shopkeeper has a faulty balance or weight is exactly similar in any situation where costs and earning are associated with a different weights/ volumes/numbers. The same scenario may also occur when a shopkeeper adulterates his goods by mixing impurities or a milk-man mixes water to the milk or the case where some goods get eaten up by rats or eggs get broken, etc. Also the same would be in the case where a meter scale expands in summers or contracts in winter.

In any of these scenarios, focus on what measure will the cost be associated with and what measure will the earnings be associated with.

E.g. 29: In summers, all metallic meter scales expand by 10%. Find the profit that a cloth trader makes in summers if he uses a metallic meter scale to measure cloth and if he sells at a mark-up of 10%.

A meter scale is 100 cms in length in its true state. In summers it expands and will measure 110 cms but it is still a meter scale i.e. its readings will not change and thus the price charged for will only be for 1 mt i.e. 100 cms. Thus if the rate at which the cloth trader purchases his goods is Rs. x per cm,

earnings of the cloth trader = 100 cms × $1.1x = 110x$, and

cost to the cloth trader = 110 cms × $x = 110x$.

Thus it’s the case of no profit no loss.

E.g. 30: A milk-man mixes water equal to 16.66% of the milk that he has. Further he sells at a mark-up of 20%. Find his profit percentage, if water is available free of cost.

Lets assume that the milk man has 6 lts of milk. Thus his cost is limited to just 6 lts of milk. But then he mixes 1 lt of water to it and thus he would be earning by selling 7 lts of the diluted milk. If the rate at which he purchases milk is Rs. x per lt,

earnings of the milk-man = 7 lts × $1.2x$ = Rs. $8.4x$, and

cost to the milk-man = 6 lts × x = Rs. $6x$.

Thus his profit percentage = 2.4/6 i.e. 2/5 i.e. 40%

Use of multiplying factor in case of faulty balances

We strongly recommend the use of multiplying factors in any scenario. This is how you can use it in case of faulty balances:

We could consider the profit percentage to be the successive effects of profit percentages due to faulty weights and profit percentage due to selling it at higher rate than rate of purchase. Thus, if f_p, f_r, f_w are the multiplying factors related to profit, increase in rate and faulty weights, then $f_p = f_r \times f_w$

The multiplying factor because of the higher rate is simply the fraction equivalent to percentage increase in rate.

The multiplying factor because of faulty weights is $\frac{\text{weight associated with earnings}}{\text{weight associated with costs}}$. From the shop-keeper's perspective, this will be $\frac{\text{reading of the balance/weight}}{\text{actual weight of goods}}$

The reason multiplying factor is being recommended is that they can take care of many factors that affect profit percentage in a single expression. Thus, if there is an increase in rate, faulty weights, adding of impurities, rats eating away some goods, etc, all of them can be thought of as successive percentage changes with the net effect being the profit percentage. Thus, $f_p = f_1 \times f_2 \times f_3 \times \ldots$, where f_1, f_2, f_3, ... are the multiplying factors due to the various factors. See following e.g. 32.

Short-cut:

If your logic is stronger than your maths and you can identify whether the shopkeeper has a profit or loss due to the faulty weights, then you could also use the fact that in case of profit, the multiplying factor has to be greater than 1 and in case of loss the multiplying factor has to be less than 1. Awareness of this can also help you arrive at the correct multiplying factor very fast.

E.g. 31: Inspite of selling at a 20% higher rate as compared to rate of purchase, a shopkeeper is able to make a profit of only 10% due to his faulty balance.

i. What will be the reading shown on the balance, when goods weighing 1 kg are kept on the balance

ii. What will be the weight of goods kept in the balance when the balance reads 1 kg

Solving this question, using multiplying factors ...

If f_p, f_r and f_w are the multiplying factors related to profit, increase in rate and faulty weights, then using $f_p = f_r \times f_w$ we have $\frac{11}{10} = \frac{6}{5} \times f_w \Rightarrow f_w = \frac{11}{12}$

Thus, $\frac{\text{reading}}{\text{actual weight}} = \frac{11}{12}$

i. $\frac{\text{reading}}{\text{1000 gms}} = \frac{11}{12} \Rightarrow$ reading = 916.66 gms

ii. $\frac{\text{1000 gms}}{\text{actual weight}} = \frac{11}{12} \Rightarrow$ actual weight = 1090.90 gms

E.g. 32: A shopkeeper purchases branded wheat from a wholesaler at a discount of 12.5% over the MRP listed. In his storage rats eat away 10% of the wheat. He mixes impurities to the extent of 10% of the wheat present. Further while selling he sells at a discount of 6.66% on the MRP. And his balance reads 1000 for 1100 gms. Find his profit/loss percentage.

Method 1:

Since 12.5% i.e. $\frac{1}{8}$ and 6.66% i.e. $\frac{1}{15}$ have to be found of the MRP consider the MRP as Rs. 120/kg. Thus his rate of purchase will be Rs. 105/kg and rate of selling will be Rs. 112/kg.

Let him purchase 100 kgs. So his cost is 100 × 105 = Rs. 10,500.

Rats will eat away 10 kgs and wheat left will be 90 kgs. He will add 10% impurities i.e. 9 kgs and thus, wheat to be sold will be 99 kgs.

But due to faulty balance an actual weight of 1100 gms will be read as 1000 gms on the balance. Thus, an actual weight of 99 kgs will be read as $\frac{10}{11} \times 99 = 90$ kgs on the balance.

Thus, his earnings = 90 × 112 = 10,080

Thus, his loss is Rs. 420 and loss percentage is $\frac{420}{10500} \times 100 = 4\%$

Method 2: Short-cut:

$f_p = \frac{8}{7} \times \frac{9}{10} \times \frac{11}{10} \times \frac{14}{15} \times \frac{10}{11} = \frac{24}{25}$. Since this is less than 1, the shopkeeper makes a loss and the loss percentage is 1/25 i.e. 4%.

Read the following to see how the multiplying factors are arrived at:

i. Discount of 12.5% on MRP while purchasing from wholesaler results in a profit to the shopkeeper. A discount of 12.5% results in a multiplying factor of $\frac{7}{8}$. But since this factor results in a profit, we have to use a multiplying factor more than 1 i.e. we have to use $\frac{8}{7}$. Please not that 12.5% discount is $1/8^{th}$ **less** and hence just because we need a multiplying factor greater than 1 does not mean we use $\left(1+\frac{1}{8}\right)$. We will have to use $\left(1-\frac{1}{8}\right)$ i.e. $\frac{7}{8}$; but since we need more than 1, a case of profit, the true calculation is division by $\frac{7}{8}$ i.e. multiplication by $\frac{8}{7}$.

2. Rats eat away 10% results in loss to the shopkeeper. 10% decrease is a multiplying factor of $\frac{9}{10}$, i.e. less than 1 and hence has to be used as $\frac{9}{10}$ itself.

3. Mixes impurities to an extent of 10% results in profit to the shopkeeper. 10% increase is a multiplying factor of $\frac{11}{10}$, i.e. more than 1 and hence has to be used as $\frac{11}{10}$ itself.

4. Sells at a discount of 6.66% on MRP results in a loss to shopkeeper. So multiplying factor to be used is $1-\frac{1}{15}$ i.e. $\frac{14}{15}$.

Balance reads 1000 for 1100, reads less for more i.e. a loss to shopkeeper. So multiplying factor used will be $\frac{10}{11}$

Note: It is wrong to assume that 'rats eating away 10%' and 'adding impurities to the extent of 10%' will cancel each other out. In terms of multiplying factors, the net effect of these two factors is $\frac{9}{10}\times\frac{11}{10}=\frac{99}{100}$ i.e. a loss of 1%. Consider that you have 100 kgs of wheat and rats eat away 10%, you will be left with 90 kgs and after mixing impurities to the extent of 10%, you will be left with 99 kgs. It does not matter whether rats eat first or impurities are added first. You will have to consider that the shop-keeper is unaware of the various factors i.e. don't add the reasoning, that since rats eat away goods, he will compensate that loss in his mind and add impurities equal to 10% of the wheat that was supposed to be present. The reality is that he purchases in bulk, 100 kgs is our assumed value, stores goods in his godown where rats eat away. When he has to add impurities, he adds to the extent of 10% of the wheat present, thinking that this percent is what cannot be noticed by customers. 10% is not of the wheat he considers to be present. If still not convinced, consider rats eating away 80% of the wheat. Would he still add 10% of the amount of wheat he had purchased or 10% of the amount of wheat present?

Exercise

24. A milk-man sells milk after mixing water to it to such a extent that water accounts for 20% of the mixture. If he sells at a mark-up rate of 10%, find his actual profit percentage.

    1. 30% 2. 32% 3. 33.33% 4. 37.5% 5. 40%

25. A shop-keeper purchases 12 dozen eggs. But 1 dozen of them are rotten and hence he has to throw them away. If he sells the remaining at a mark-up of 8.33%, find his profit percentage.

    1. No profit No loss 2. 7.28% profit 3. 4.16 profit
    4. 1.44% loss 5. 0.7% loss

26. A shop-keeper, unaware that his balance reads 900 gms for 1000 gms, sells goods at a mark-up of 20%. Find his actual profit percentage.

    1. 8% 2. 10% 3. 11.11% 4. 12% 5. 12.5%

27. A shop-keeper purchases his goods from a wholesaler whose balance reads 1100 gms for 1000 gms. The shop-keeper then sells his wares after marking up the prices by 10%. Find his overall profit or loss percentage.

    1. 0% 2. 1% loss 3. 1% profit 4. 215 profit 5. 19% loss

28. A trader uses a faulty 1 kg weight such that even after selling his goods at a discounted price of 10% of his rate of purchase, he manages to make a profit of 10%. Find the actual weight that the faulty 1 kg weight measures.

    1. 1100 gms 2. 900 gms 3. 888.88 gms 4. 800 gms 5. 818.8 gms

29. In summers, the meter scale of a cloth merchant expands by 25%. By what percentage should he mark up the cloth so that even after giving a 10% discount on the list price, the merchant still makes a profit of 8%?

    1. 20% 2. 25% 3. 33.33% 4. 40% 5. 50%

30. A trader mixes 20% impurity in the rice that he is selling and he marks up the price further by 10%. Yet he makes no profit because he gives Y litres extra on every X litres bought. What is the ratio of Y and X?

    1. 25 : 33 2. 33 : 25 3. 8 : 25 4. 8 : 33 5. Cannot be determined

Puzzles

1. A bookstall owner marks a book purchased at Rs. 300 at Rs. 400. A tourist purchases the book and tenders a Rs. 500 note. Not having change, the bookstall owner gives the Rs. 500 note to his neighbour and gets change of which he gives Rs. 100 back to the tourist. Next day, the neighbour informs him that the Rs. 500 note was a counterfeit one and takes a genuine Rs. 500 from the bookstall owner. Find the loss to the bookstall owner.

2. *A* purchases a watch at Rs. 500. After some day he sells it to *B* at Rs. 600. After some more day, *B* sells the watch back to *A* at Rs. 700. Yet after some more day, *A* sells the watch back to *B* at Rs. 800. Who among *A* or *B* is at a profit and who at a loss? And of how much?

Simple Interest, Compound Interest

When we part with our money to keep it in the bank, the bank pays us an 'interest' to do so. Similarly when we borrow money, we have to pay an 'interest'. The interest received or paid depends on three factors:

Principal (P)

The amount of money that is loaned out or is borrowed is called the Principal. Interest would obviously depend on the principal. Higher is the amount, more is the interest and lower the amount, lesser is the interest.

Time Period (t)

Again it should be obvious that if the money is loaned or borrowed for a longer time period, more interest will be charged and for a shorter time period, lesser will be the interest.

Rate of Interest (r)

There is a third factor which determines the Interest. This is the Rate of Interest, expressed as a percentage. This is a pre-determined amount that the giver and the receiver agree to and it specifies the amount of interest charged per Rs. 100 of the principal for a specified time period. Thus if the rate of interest agreed is 8% per annum, it means that on every Rs. 100 of the principal amount, the interest will Rs. 8 per year.

Further, there are two ways in which an Interest can be charged. For each of the year the money is borrowed/loaned, Interest could be charged just on the principal amount loaned/borrowed (as done in Simple Interest) or for each of the year the interest could be charged on the amount outstanding i.e. principal amount and the accrued interest of the previous years (as done in Compound Interest). These two methods are discussed in details in the following sections.

Simple Interest

In this method the interest charged per year is calculated only on the principal. What this means is that if Rs. 1000 will be loaned/borrowed, the interest will be charged only on Rs. 1000 even in successive years.

Thus if the rate is r% per annum, then Simple Interest per year = r% of P and thus we get the following formula for *SI*, which most of us are already acquainted with.

$$\text{SI} = \frac{\text{P} \times r \times t}{100}$$

Amount returnable = Principal + SI

E.g. 1: If A deposits Rs. 20,000 in a bank for 3 years at a rate of 10%, what is the simple interest he will get at the end of the period?

Here P = 20,000, r = 10%, t = 3 yrs

Hence SI = $\frac{20,000 \times 10 \times 3}{100}$ = Rs. 6,000

E.g. 2: Sagar borrows Rs. 50,000 from a bank for 5 years. What is the rate of simple interest charged by the bank if after 5 years Sagar had to pay Rs. 66,000 to the bank?

Here 66,000 is the total outstanding amount which is Principal + SI. Since principal is given as Rs. 50,000, we get SI = 66,000 – 50,000 = Rs. 16,000

Using the above formula, $16,000 = \frac{50,000 \times r \times 5}{100} \Rightarrow r = \frac{32}{5}$ = 6.4%

E.g. 3: After how many years would an amount double itself at 15% rate of simple interest?

We don't have the principal here, we can denote it by P. If the amount doubles then obviously the SI would be equal to the principal.

So we have $P = \frac{P \times 15 \times t}{100} \Rightarrow t = \frac{100}{15}$ = 6.667 years

E.g. 4: The rate of simple interest of a bank is 10%. Akash was taking a loan of a huge amount for one year so the bank agreed to give him the loan at a rate of 8%. They reasoned that even with 8% they will get twice the interest that they would have got had they given out a loan of Rs. 40,00,000 at 10% for a year. What is the amount of Akash's loan?

If P is the loan taken by Akash, writing the relation comparing the two simple interests,

$$\frac{P \times 8 \times 1}{100} = 2 \times \frac{40,00,000 \times 10 \times 1}{100} \Rightarrow P = 1,00,00,000$$

Compound Interest

In the case of Compound Interest, after a fixed time intervals (again pre-determined and agreed to by the giver and receiver) the interest amount is calculated on the principal and this interest amount is added to the principal to determine the current outstanding amount. Then during the next time interval, Interest is charged on the current outstanding amount and not on the principal. Again this interest is added to find the outstanding amount. This process is continued for successive time intervals.

Explanation of the above using a numeric example:

Consider a loan of Rs. 1000 is taken with compound interest being charged at a rate of 10% p.a. Read the following table row-wise.

Year	Amount outstanding at start of year	Interest charged this year	Amount outstanding at end of year
1st year	1,000	100	1,100
2nd year	1,100	110	1,210
3rd year	1210	121	1,331
4th year	1,331	133.1	1,464.1

Compounding

Compounding is the process of adding the interest accrued to the principal. In the above example the interest was added back to the principal at the end of every year. Hence it is called annual compounding.

However note that the interest of Rs. 100 earned in the 1st year is not earned on the 365th day of the year. It is continuously being earned throughout the year. Thus, after 6 months, the interest earned would have been Rs. 50. However one waits for the entire year to pass before adding it back to the principal, because it is a case of annual compounding.

Had the giver and receiver agreed on a 6-months compounding, called semi-annual compounding, the time period would have been measured in spans of 6 months and the principal would be increased by the interest accrued every six months. In that case the table would look like:

Time period	Amount outstanding at start of time period	Interest charged this time period	Amount outstanding at end of time period
First 6 months	1000	100/2 = 50	1050
Second 6 months i.e. 1 yr from start	1050	105/2 = 52.5	1102.5
Third 6 months	1102.5	110.25/2 = 55.125	1157.625
Fourth 6 months i.e. 2 year from start	1157.625	115.76/2= 57.88	1215.505

Compounding can be any time period – annually, semi-annually, quarterly, monthly, daily. There is also a case of continuous compounding where the time period is considered very very small, i.e. interest earned every moment is being added back to the principal and interest earned in next time period being found on this increased principal. But this case of continuous compounding is out of scope for our purposes.

The formula in case of CI is: $A = P\left(1+\frac{r}{100}\right)^n$, where *r* is the agreed rate for a predefined interval of time and *n* is the number of time periods the loan is taken or amount is deposited for.

This formula gives us the final outstanding amount and NOT the Compound Interest.

Compound Interest = Amount – Principal

E.g. 5: A bank charges a rate of interest of 10% compounded annually. What is the total amount to be paid on a loan of Rs. 36000 for 2 yrs?

Using the above given formula, $A = 36,000\left(1+\frac{10}{100}\right)^2 = 36,000 \times \frac{121}{100}$

= Rs. 43,560

E.g. 6: A man takes a loan of Rs. 1,00,000 for two years at compound interest. If he has to return Rs. 1,10,250, find the rate of interest charged.

Using the formula we have $1,10,250 = 1,00,000\left(1+\frac{r}{100}\right)^2$

Doing this calculation and finding the square root is going to be a cumbersome process. The better alternative is to make an educated guess and then confirm this by squaring as follows:

The man pays an interest of Rs. 10,250 on a principal of 1,00,000. Thus the interest paid in two years is a little over 10%. A good guess would be that the interest rate per annum is 5%. Assuming it is 5% and finding the square of 1.05 we see that $1.05^2 = 1.1025$.

Thus we can confirm that the interest rate was indeed 5%.

E.g. 7: Manu lends a sum of money to his friend at interest such that the amount triples after 5 years when compounded annually. In how many years would the amount payable back become 9 times the sum loaned?

Here Amount becomes thrice the Principal. So,

$$3P = P\left(1+\frac{r}{100}\right)^5 \Rightarrow \left(1+\frac{r}{100}\right)^5 = 3$$

The question requires us to find *n* when $9P = P\left(1+\frac{r}{100}\right)^n$ i.e. when

$$\left(1+\frac{r}{100}\right)^n = 9$$

Denoting $\left(1+\frac{r}{100}\right)$ as f, we have $f^5 = 3$ and we want to find $f^n = 9$.

Obviously since 9 is square of 3, we will have $f^{10} = 9$. Thus the amount will become 9 times the principal in 10 years.

Alternate method: In the case of compound interest, it is a fresh beginning after every compounding. Thus after 5 years, when the principal triples, it is as good as a fresh beginning. In the next 5 years this amount, 3P, will again triple i.e. will become 9P. Thus, in a total of 10 years, the amount will become 9 times.

$\left(1+\frac{r}{100}\right)$ as a multiplying factor

With $r = 10\%$, $\left(1+\frac{r}{100}\right)$ will become 1.1 or $\frac{11}{10}$

With $r = 20\%$, $\left(1+\frac{r}{100}\right)$ will become 1.2 or $\frac{6}{5}$

The term $\left(1+\frac{r}{100}\right)$ is nothing but the multiplying factor equivalent to a r % increase. Thus, just to save space and to think on lines of a percentage increase, we shall denote $\left(1+\frac{r}{100}\right)$ as f.

E.g. 8: A man loans out Rs. 50,000 at a rate of 8% simple interest for 2 years and another Rs. 50,000 at the same rate compounded annually for 2 years. Is there any difference in the two amounts he gets back after 2 years?

Lets first calculate the total amount in the case of Simple Interest

$$SI = \frac{50,000 \times 8 \times 2}{100} = \text{Rs. } 8000$$

Now, let's calculate the total amount in the case of Compound Interest

$A = 50,000 \times 1.08^2 = 50,000 \times 1.1664 = \text{Rs. } 58,320$

This is the amount and not the CI. The CI = 58,320 – 50,000 = Rs. 8,320

Hence there is a difference of Rs. 320 in the two amounts he gets back

E.g. 9: If interest is compounded semi annually at 10% per annum, what will be the total amount at the end of 2 years if principal is Rs. 15000.

Non-Annual Compounding

The formula for non-annual compounding remains the same, $A = P\left(1+\frac{r}{100}\right)^n$. The only difference being that here r will be the rate of interest per 'compounding period'. If in a question, it is given that compounding happens half yearly but r is given as the rate per annum, then we will need to divide r by 2 to get half yearly rate of interest. Similarly for quarterly compounding, if r is given as the rate per annum, then we will need to divide r by 4 to get quarterly rate of interest.

Also remember that t here is the number of 'compounding periods'. So if the interest is compounded half yearly, and number of years, n is given, the number of time periods will be equal to $2 \times n$.

Since the interest is compounded semi annually and rate of interest is per annum, the semi annual rate of interest will be 10/2 = 5%. Also t will be $2 \times 2 = 4$.

$$\text{So A} = 15{,}000\left(1+\frac{5}{100}\right)^4 = 15{,}000 \times \left(\frac{21}{20}\right)^4 = \text{Rs. } 18{,}233$$

E.g. 10: If a bank offers two schemes (i) Annual compounding at 11% (ii) Semi-annual compounding at 10% p.a., which of the two is a better scheme for depositors who want to deposit their money for two years?

Let's assume principal to 100.

Case (i) In case of annual compounding $r = 11\%$, $t = 2$

So A = $100 \times 1.1^2 = 121$.

Case (ii) In case of Semi-annual compounding $r = 10/2 = 5\%$, $t = 4$

So A = $100 \times 1.05^4 = 100 \times 1.1025^2$.

One should not calculate and find the square. Remember we just need to compare and it is obvious that 1.1025^2 will surely be greater than 1.1^2.

Hence Case (ii) of semi-annual compounding is better for the depositors.

It should be obvious that people seeking loans would prefer Case (i) since they will need to pay less interest at the end of the year.

Note: Population, Appreciation and Depreciation are generally calculated at compound rate of interest unless otherwise stated. In most of the other cases unless otherwise stated, we will assume simple rate of interest.

Unless otherwise stated, compound interest will be compounded annually.

Exercise

1. Ajay takes a loan of Rs. 30,000 from a bank for 8 years at 6.5% rate of simple interest. He then loans out Rs. 20,000 for 8 years at 7.5% rate of simple interest. He could loan out the balance only at 5.5% for 8 years. In the entire transaction, did Ajay make or lose money and how much?

   1. Gain 400   2. Gain 800   3. No gain, No loss
   4. Loss 400   5. Loss 800

2. Vijay took part of Rs. 10,000 loan at 4% and the rest at 6%. If he pays a total interest of Rs. 900 in two years, find the amount taken on loan at 4%. The interest rate charged by the bank is Simple Interest.

   1. Rs 8000   2. Rs 7500   3. Rs 7000   4. Rs 6000   5. Rs 5000

3. The underworld don Chhota Pappu loans money to people at simple interest. He charges a certain rate of interest for the first year. Next year he doubles the initial rate of interest on the amount. Third year he triples the initial rate of interest and so on... A man took an amount of Rs. 9000 and after 3 yrs paid back an amount of Rs. 15000 back. What was the rate of interest in the first year?

   1. 9%   2. 9.09%   3. 10%   4. 11.11%   5. 12.5%

4. I needed Rs. 1,20,000 to buy a Plasma TV and hence I borrowed Rs. 75,000 from Vani and the rest from Vivek. Vani and Vivek charge me a rate of interest such that the interest amount payable to both of them is the same. If in all I re-pay them a total of Rs. 1,50,000 at the end of 2 years, what is the rate of interest charged per annum by the two?

   1. 10%,20%   2. 12.5%, 20%   3. 10%,16.66%   4. 12.5%, 16.66%5. 16.66, 20%

5. At a certain rate of simple interest, a principal becomes three times in 15 years. In how many years will the principal amount become nine times?

   1. 45 years   2. 30 years   3. 60 years   4. 75 years   5. None of these

6. What approximate rate per annum of simple interest would yield the same amount as that got at compound interest rate of 20% p.a. when the same principal is kept for three years in both the cases.

   1. 72.8%   2. 64%   3. 32%   4. 21.33%   5. 24.23%

7. What will the approximate amount be after 3 years if I deposit Rs. 5000 in a bank which offers me a rate of interest of 5%, compounded annually?

   1. Rs 5750   2. Rs 5760   3. Rs 5770   4. Rs 570   5. Rs 5790

8. The population of a city grows at a rate of 5% per annum. If in 2006 its population is 18,52,200, what was its population in 2004?

   1. 12,60,000 2. 13,60,000 3. 15,60,000 4. 16,00,000 5. 16,80,000

9. The property prices appreciate at a rate of 7% per annum. I bought a house in the year 2003 which had cost me Rs. 10,00,000 at that time. What will be its approximate cost three years later?

   1. 12,60,000 2. 12,23,000 3. 12,25,000 4. 12,27,000 5. 12,29,000

10. I bought an Astra two years back. Its value depreciated by 9% every year. If at present its value is Rs. 9,10,910, at what cost had I bought it?

    1. 10,91,910 2. 10,9,190 3. 10,00,000 4. 11,00,000 5. 12,00,000

11. On investing Rs. 5000 in a bank, you will get back Rs. 5671 in 2 years. What is the approximate compound rate of interest?

    1. 5% 2. 5.5% 3. 6% 4. 6.5% 5. 7%

12. A bank offers a rate of interest of 16% per annum, compounded semi annually. What will be the approximate interest generated on an amount of Rs. 10,000 kept for 2 years?

    1. Rs 3600 2. Rs 3605 3. Rs 3610 4. Rs 3615 5. Rs 3620

13. Which of the following two schemes is more beneficial to a depositor with a two year investment horizon?

    (i) Rate of interest 6% compounded annually.

    (ii) Rate of interest 5% compounded semi-annually.

    1. (i) 2. (ii)

    3. Both are same 4. Depends on principal amount

14. I deposited an amount of Rs. 20,000 at a 10% p.a. rate of interest compounded quarterly. What is the approximate amount due to me at the end of one year?

    1. Rs 22,000 2. Rs 22,025 3. Rs 22,050 4. Rs 22,075 5. Rs 22,100

15. A cooperative bank lent Rs.4000 to Anand at a certain rate of simple interest and Rs.5000 to Milind at ½% more than that of Anand. After 2 years the bank received Rs. 860 as total interest from Anand and Milind. Find the rate of interest per annum at which the amount was lent to Milind.

    1. 5% 2. 4.5% 3. 5.5% 4. 4% 5. None of these

C. I.: Case of Successive Percentage Changes

Rs. 1000 is kept in a bank at compound interest of 10% p.a. The following diagram depicts the growth of money on a year on year basis:

Year 1 Year 2 Year 3

$$\text{Rs. } 1000 \xrightarrow{10\%\text{ inc}} \text{Rs. } 1100 \xrightarrow{10\%\text{ inc}} \text{Rs. } 1210 \xrightarrow{10\%\text{ inc}} \text{Rs. } 1331$$

Two successive increases of 10%
i.e. a net increase of
$10 + 10 + \frac{10 \times 10}{100} = 21\%$

Three successive increases of 10%
i.e. a net increase of
$21 + 10 + \frac{21 \times 10}{100} = 33.1\%$

The above representation should make it very clear that compound interest is a case of successive percentage increases of r% every year. This fact could be used effectively in certain situations as explained in the following example.

E.g. 11: What rate per annum of simple interest would yield the same amount as that got at compound interest rate of 25% p.a. when the same principal is kept for three years in both the cases.

Using formula, the amount at 25% compound rate after 3 years is $P \times 1.25^3$. Finding 1.25^3 will be slightly difficult.

Alternately, in compound interest the total percentage increase over the three years will be the net effect of 25% & 25% & 25% increase.

25% & 25% increase is equivalent to 56.25% increase.

Thus the net increase is $25 + 56.25 + \frac{25 \times 56.25}{100}$ = 81.25 + 14.0625 = 95.3125%

For the same percentage increase over three years in the case of simple interest, since each year we get the same percentage increase, the rate should be $\frac{95.3125}{3}$ = 31.7708 per year.

E.g. 12: The population of a city grows at a rate of 12.5% per annum. If in 2006 its population is 7,29,000, what was its population in 2003?

A 12.5% increase corresponds to a multiplying factor of $\frac{9}{8}$

Thus, 7,29,000 = $x \times \frac{9}{8} \times \frac{9}{8} \times \frac{9}{8}$, where x is the population in year 2003.

Since $9^3 = 729$ and $8^3 = 512$, we can find x = 5,12,000

This approach is better than finding 1.125^3

C.I.: Case of Interest on Interest

In competitive exams there are many questions involving comparing simple interest and compound interest in the first two years or comparing the compound interest in two successive years. So let's look at this more thoroughly:

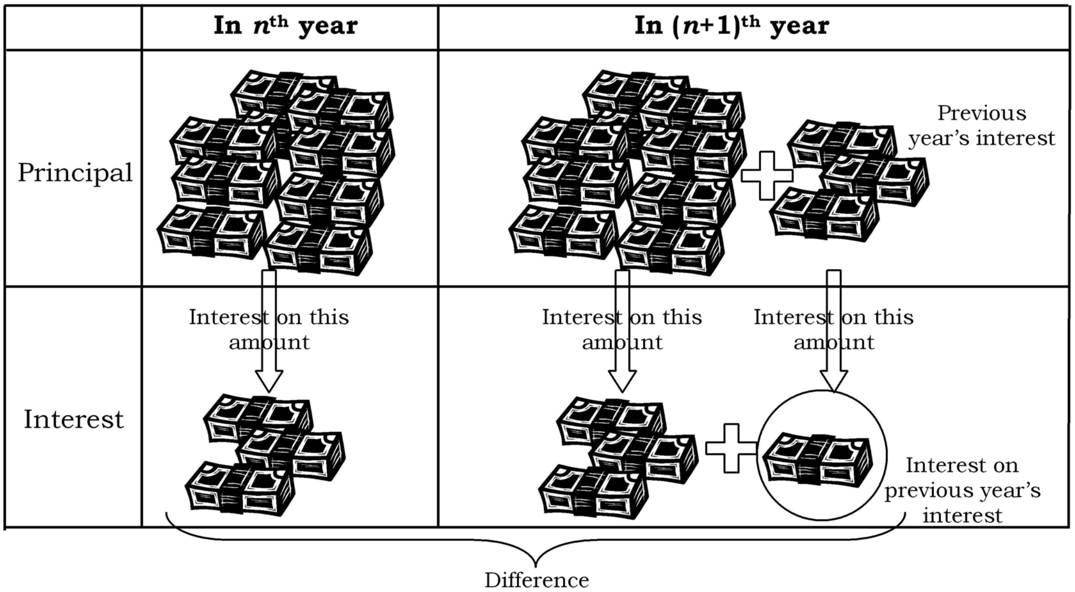
Compound Interest In Successive Years

In the case of compound interest the amount on which interest is calculated, keeps changing. The first year, it is equal to the principal, next year it is equal to principal plus last year's interest, in the year after that, interest is calculated on principal plus the interest earned in the previous two years and so on. This is what we mean by compound interest. The interest gets compounded (added to the principal) every year. Every subsequent year, the interest calculated for that year, will be more than the interest calculated for the previous year because the amount on which interest is calculated would be more than the amount of the previous year.

The following figure compares the interest earned in two successive years, n^{th} and $(n + 1)^{th}$. The picture is self sufficient to understand that, in any year, one would receive as much interest as earned in the previous year PLUS one would earn interest on the previous year's interest.

And this should be logical because the previous year's interest gets added to the amount at end of previous year.

Thus, **difference in compound interest earned in two successive years is equal to the interest on the interest of first of these years.**



While the image does not make it very clear (the table in the following box will), a little thought will make it obvious that since the difference is interest on previous years interest, **the ratio of compound interest earned in two successive years is (1 + *r* %) : 1**

Mathematically ...

Let us say a sum is kept at r% compound interest.

Year	Principal at start of year	Interest in year	Amount at end of year
$(n-1)^{th}$	Unknown	Unknown	x, say
n^{th}	x	$CI_n = r\%$ of x	$x + CI_n$
$(n+1)^{th}$	$x + CI_n$	$CI_{n+1} = r\%$ of $(x + CI_n)$ $= r\%$ of $x + r\%$ of CI_n $= CI_n + r\%$ of CI_n	

Two important conclusions from the above

1. The difference $CI_{n+1} - CI_n = r\%$ of CI_n i.e. difference between compound interest earned in two successive years is the interest earned on interest of first of these years.

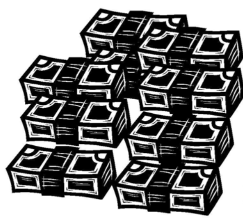
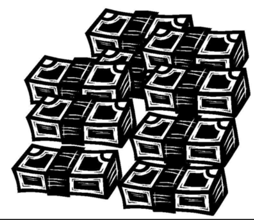
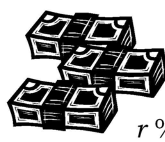
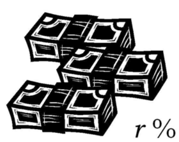
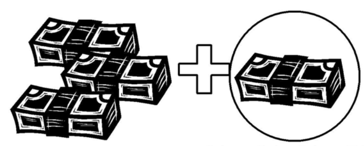
2. $\frac{CI_{n+1}}{CI_n} = \frac{1+r\%}{1}$

E.g. 13: If the ratio of CI in the 7th and 8th year is 10 : 11, find the rate of interest being offered.

Rather than use the formula for the ratio, the CI in the two years can be assumed as $10k$ and $11k$. Needless to say the extra k interest earned is on $10k$ and hence the rate of interest is 1/10 i.e. 10%

Difference Between SI And CI In First Two Years

Consider a principal kept at simple interest at certain rate. Also consider the same amount kept at same rate but this time at compound interest (annually). Now let us compare the two cases over the first two years.

	Simple Interest	**Compound Interest**
Principal	 P	 P
Interest 1st Year	 r % of P	 r % of P
Interest 2nd Year	 r % of P	 r % of P r % of (r % of P)

Difference between total CI and total SI earned in first two years = Interest on the first years interest i.e. r % of I_1. But I_1 = r % of P. Thus, difference between the total CI and SI earned in first two years = r % of (r % of P) i.e. $\frac{r^2}{100}$ % of P

Ratio of CI and SI earned in first two years:

$$\frac{\text{CI earned in first 2 years}}{\text{SI earned in firsts 2 years}} = \frac{2\times(r\%\text{ of P}) + r\%\text{ of }(r\%\text{ of P})}{2\times(r\%\text{ of P})} = \frac{2+r\%}{2}$$

E.g. 14: What is the difference between SI and CI accrued in the 2nd year (only in 2nd year, not 1st and 2nd combined) on a principal of Rs. 1000 at 8% interest?

Since the interest earned in 1st year is same whether it is SI or CI, the difference between the total CI and SI earned in first two years is actually the difference between the CI and SI of the 2nd year itself. Thus required difference = $\frac{8^2}{100 \times 100} \times 1000$ = Rs. 6.40

E.g. 15: If the rate of interest is 15%, what is the ratio of the total CI earned in first 2 years to the total SI earned in first two years if the principal kept is same?

It's a straightforward application of formula found earlier i.e.
$\frac{2+0.15}{2} = \frac{2.15}{2}$ i.e. 43 : 40

E.g. 16: If the ratio of the total compound interest earned in firsts two years to the total simple interest earned in first two years is 11 : 10, find the rate of interest, assuming it and the principal to be the same in the two cases.

Rather than using the formula, one could also think on following lines:

Let total SI earned in 2 years be $10k$. Thus, in the first year the SI earned, as well as the CI earned will be $5k$. Thus, in the second year, the CI earned will be $11k - 5k = 6k$.

In the second year, the CI earned is more than the first years CI of $5k$ by k. Thus, rate of interest is $k/5k$ i.e. 1/5 i.e. 20%

Exercise

16. The CI earned in the 7^{th} year is Rs. 500. If the rate of interest is 15%, find the compound interest earned in the 8^{th} year.

    1. Rs 500 2. Rs 525 3. Rs 550 4. Rs 575 5. Rs 600

17. The compound interest earned in the 3^{rd} and 4^{th} year is Rs. 450 and Rs. 500. Find the rate of interest.

    1. 9.09% 2. 10% 3. 11.11% 4. 12.5% 5. 15%

18. At a compound interest rate of 10%, the compound interest earned in the 8^{th} year is Rs. 484. Find the compound interest earned in the 6^{th} year.

    1. Rs 360 2. Rs 400 3. Rs 440 4. Rs 480 5. Cannot be determined

19. If the rate of compound interest is 12.5%, find the ratio of compound interest earned in the 24^{th} year and that earned in the 25^{th} year.

    1. 8 : 9 2. 9 : 8 3. 24 : 25 4. 192 : 225 5. 25 : 27

20. If the ratio of compound interest earned in the n^{th} and the $(n+1)^{th}$ year is 15 : 16, find the rate of interest.

    1. 12.5 % 2. 13.33% 3. 6.66% 4. 6.25% 5. Depends on value of n

21. The difference between the compound interest and simple interest on a certain sum at 10% per annum for 2 years is Rs. 631. Find the sum.

    1. Rs 6,310 2. Rs 63,100 3. Rs 6,31,000 4. Rs 63,10,000 5. Cannot be determined

22. The difference between the compound interest and simple interest accrued on an amount of Rs. 18,000 in 2 years was Rs. 405. Find the rate of interest, if it is same in the case of simple and compound interest.

    1. 10% 2. 12.5% 3. 15% 4. 17.5% 5. 20%

23. I kept Rs. 20,000 at 5% rate of simple interest for two years. Find the difference in interest earned if I had kept the same amount for same years at same rate but at compound interest.

    1. Rs 1 2. Rs 2 3. Rs 5 4. Rs 20 5. Rs 50

24. If the rate of interest in case of both compound and simple interest is 8.33%, find the ratio of the compound interest and simple interest earned in first 2 years on the same principal.

    1. 25 : 24 2. 24 : 25 3. 11 : 12 4. 12 : 1 5. None of these

CAT Questions

1. *[CAT 2006]* The length, breadth and height of a room are in the ratio 3 : 2 : 1. If the breadth and height are halved while the length is doubled, then the total area of the four walls of the room will

   (1) remain the same (2) decrease by 13.64% (3) decrease by 15%
   (4) decrease by 18.75% (5) decrease by 30%

2. *[CAT 2004]* The question is followed by two statements, A and B. Mark your answer as
   (1) if the question can be answered by using one of the statements alone but not by using the other statement alone
   (2) if the questions can be answer by using either of the statements alone.
   (3) if the question can be answered by using both statements together but not by either statements alone.
   (4) if the question cannot be answered on the basis of the two statements.

   Zakib spends 30% of his income on his children's education, 20% on recreation and 10% on healthcare. The corresponding percentages for Supriyo are 40%, 25% and 13%. Who spends more on children's education?

   A: Zakib spends more on recreation than Supriyo
   B: Supriyo spends more on healthcare than Zakib

3. *[CAT 2003 – Leaked]* Let A and B be two solid spheres such that the surface area of B is 300% higher than the surface area of A. The volume of A is found to be k% lower than the volume of B. The value of k must be___.

   (1) 85.5 (2) 92.5 (3) 90.5 (4) 87.5

4. *[CAT 2003 – Leaked]* At the end of year 1998, Shepard bought nine dozen goats. Henceforth, every year he added p% of the goats at the beginning of the year and sold q% of the goats at the end of the year where $p > 0$ and $q > 0$. If Shepard had nine dozen goats at the end of year 2002, after making the sales for that year, which of the following is true?

   (1) $p = q$ (2) $p < q$ (3) $p > q$ (4) $p = \frac{q}{2}$

5. *[CAT 2001]* A college has raised 75% of the amount it needs for a new building by receiving an average donation of Rs. 600 from the people already solicited. The people already solicited represent 60% of the people the college will ask for donations. If the college is to raise exactly the amount needed for the new building, what should be the average donation from the remaining people to be solicited?

   (1) Rs. 300 (2) Rs. 250 (3) Rs. 400 (4) Rs. 500

6. *[CAT 2001]* Fresh grapes contain 90% water by weight while dried grapes contain 20% water by weight. What is the weight of dry grapes available from 20 kg of fresh grapes?

   (1) 2 kg (2) 2.4 kg (3) 2.5 kg (4) None of these

7. *[CAT 2001]* The owner of an art shop conducts his business in the following manner: Every once in a while he raises his prices by X%, then a while later he reduces all the new prices by X%. After one such up-down cycle, the price of a painting decreased by Rs. 441. After a second up-down cycle the painting was sold for Rs. 1944.81. What was the original price of the painting?

   (1) 2756.25 (2) 2256.25 (3) 2500 (4) 2000

8. *[CAT 2000]* The table below shows the age-wise distribution of the population of Reposia. The number of people aged below 35 years is 400 million.

Age group	Percentages
Below 15 years	30.00
15 - 24	17.75
25 - 34	17.00
35 - 44	14.50
45 - 54	12.50
55 - 64	7.10
65 and above	1.15

If the ratio of females to males in the 'below 15 years' age group is 0.96, then what is the number of females (in millions) in that age group?

(1) 82.8 (2) 90.8 (3) 80.0 (4) 90.0

9. *[CAT 1999]* 40 percent of the employees of a certain company are men, and 75 percent of the men earn more than Rs. 25,000 per year. If 45 percent of the company's employees earn more than Rs.25,000 per year, what fraction of the women employed by the company earn Rs. 25,000 per year or less?

(1) $\frac{2}{11}$ (2) $\frac{1}{4}$ (3) $\frac{1}{3}$ (4) $\frac{3}{4}$

Averages & Weighted Averages

Simple Average or Arithmetic Mean

Whenever we speak of average, we understand that

$$\text{Average} = \frac{\text{Sum of all observations}}{\text{Number of observations}}$$

This is more specifically called the Arithmetic Mean and would differ in interpretation and applications from Weighted Average as we will learn ahead. But for the moment it hardly matters if we use either term – Average or Arithmetic Mean.

In the relation, there are three terms – average, A; sum of observations, S; and the number of observations, n. Given any two we should be very quick at finding the third. The most common data is 'Average of n observation is A' and it should immediately be processed as S = A × n

E.g. 1: The average weight of 10 members was 70 kgs. Find the average weight if an eleventh guy weighing 81 kgs joins the group.

Sum of weights of the 10 members = 70 × 10 = 700

Thus, sum of the weights after the eleventh guy joins the group = 700 + 81 = 781.

Thus new average = 781/11 = 71.

Average as an equal distribution

While the above is straight-forward, this method may necessitate the use of pencil work if the numbers are not very manageable, as in the next example. So, an oral approach for the above is being explained, which is also far more intellectually stimulating.

Considering that the eleventh guy also weighs same as the earlier average i.e. 70 kgs, we now have 11 extra kgs to be distributed equally among 11 students. Thus each will receive 1 kg and the new average is 71 kgs.

E.g. 2: The average amount that each student has is Rs. 167.75. When a student having Rs. 183.45 joins the group, the new average that each student has is now Rs. 170.89. Find the number of people originally in the group.

Using the formula, if there were n students earlier, then we would have

$167.75 \times n + 183.45 = 170.89 \times (n + 1)$ i.e. $3.14n = 12.56$ i.e. $n = 4$.

Short-cut

Using the "intellectually stimulating" method: Considering that the new student also has Rs. 167.75, there is now Rs. 183.45 – 167.75 = Rs. 15.70 that is distributed among all members (n + 1) and each member receives Rs. 170.89 – 167.75 = Rs. 3.14. Thus there are in all 5 members after the new student joins and there were 4 students originally in the group.

E.g. 3: The average age of a class of 24 students and 1 teacher is 15 years. If the teacher is not considered, the average age of the students is 14 years. What is the age of the teacher?

The sum of the ages of the teacher and the students is 25 × 15 = 375 and the sum of the ages of the students is 24 × 14 = 336. The difference between these, 39, will be the age of the teacher.

Again the above approach may appear very easy and you may not feel the need for any logical interpretation or alternate method. But in some scenarios, the numbers may make the logical way to solve the question more easier and the above process more calculation oriented.

Logical Solution

It is best to assume all the people as having money. In that case, the solution appears far more easier. Thus, 25 students on an average had Rs. 15 each. One of them, the teacher, left and the average amount with each of the other 24 decreased by 1 i.e. the teacher took away Rs. 1 × 24 = Rs. 24 with him. Why did he take it away? He must have given it to the students to average out (he and others having Rs. 15 each) and thus, the actual amount with him would have been 15 + 24 = 39.

E.g. 4: While analyzing my scores across 8 mock CATs that I had taken, I realized that the average marks when the worst seven scores is considered, is 5 less than the average marks when the best seven scores are considered. What is the difference between the highest and the lowest scores?

If the marks are M_1, M_2, M_3, ..., M_8 in increasing order, we have

$$\frac{M_1 + M_2 + M_3 + \ldots\ldots + M_7}{7} = A \quad \ldots\ldots\text{(i)}$$

$$\frac{M_2 + M_3 + M_4 + \ldots\ldots + M_8}{7} = A + 5 \quad \ldots\ldots\text{(ii)}$$

(ii) – (i) $\quad \dfrac{M_8 - M_1}{7} = 5 \Rightarrow M_8 - M_1 = 35$

Alternately instead of considering so many variables, we could have consider the total of all the 8 scores as T and the best and worst scores as B and W, the following essentially is the same as above:

$$\frac{\text{T} - \text{W}}{7} - \frac{\text{T} - \text{B}}{7} = 5 \Rightarrow \frac{\text{B} - \text{W}}{7} = 5 \text{ i.e. B – W = 35.}$$

Logical Interpretation

Consider the two scenarios, once when the 7 best scores are considered and once when the 7 worst scores are considered. In both the cases, the sum is divided by 7.

In the two cases, the only difference is that the worst score is removed and replaced by the best score. This causes the average to increase by 5.

Since 7 is the divisor and the average increases by 5, the sum should have increased by 35 when the worst score was replaced by the best score i.e. the difference between the best and the worst score is 35.

There would be many such instances when we would know the change in the averages and from this we can be deduce the change in the sum.

Couple of questions in CAT on this area has been about averages of an arithmetic series, specifically series of consecutive natural numbers.

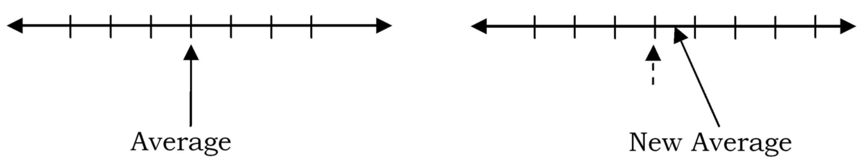
An AP is a series of numbers such that the difference between consecutive terms of the series is the same e.g. 3, 7, 11, 15, 19, ... or 2, 5, 8, 11, 14, ...

The average of an AP is the average of the first and the last term. It is also equal to the middle term of the series (if the number of terms of the series is even, then the average of the two middle terms of the series)

One of the question in CAT was similar to: If the average of a set of n consecutive natural numbers is 17.5. What is the average of the set of $n + 5$ consecutive natural numbers formed by including the next (following) 5 natural number also in the earlier set?

This example is given to explain that when we have a set of consecutive natural numbers, the number lying in the middle is the average of the set.

Thus the average of the following seven consecutive numbers as highlighted in the number line below is the middle number. If the next natural number is also included in the set, the average will increase by 0.5, as seen from the following visual:



If the above is clearly understood, when we include the next five natural numbers in any set, the average of the set will increase by 2.5 and thus the answer to the question in this example is 17.5 + 2.5 = 20.

How many of you thought that the answer would be "cannnot be determined"? Make sure that you take numbers and check the above answer. Thus say the original set was the numbers 17 and 18 (with average being 17.5). The average of 17, 18, 19, 20, 21, 22, 23 is 20. If the original set was 16, 17, 18, 19 (average has to be 17.5), we would now need to find the average of 16, 17, 18, 19, 20, 21, 22, 23, 24 which is again 20.

E.g. 5: A set contains the first 20 natural numbers, except one of them. If the average of the set is $10\frac{13}{19}$, find the number which is absent.

While one can solve this question by finding the difference between the sum of the first twenty natural numbers, $\frac{20\times 21}{2}$, and the sum of the numbers in the set, $\left(10\frac{13}{19}\right)\times 19$, learn the following approach more keenly because there can be many such difficult questions where working in the above manner many not be quite feasible.

The set of first twenty natural numbers has an average of 10.5 (average of the middle two, 10 & 11) and when one number is absent the average of the remaining 19 numbers becomes $10\frac{13}{19}$

Considering 20 friends, each having Rs. 10.5 and one of the friends leaving and the average amount with the other 19 friends increases by $\frac{13}{19}-\frac{1}{2}=\frac{7}{38}$

Thus the total amount received by these 19 numbers from the friend who left the group is $\frac{7}{38}\times 19$ = 3.5. This amount must have been taken by the friend to reach the average level of 10.5.

And hence the number not being present has to be 10.5 – 3.5 = 7.

Exercise

1. The average sales for the months January to March is 40, the average sales for the months March to June is 50 and the average of the months January to June is 45. Find the sales in the month of March.

   1. 30 2. 40 3. 45 4. 50 5. 60

2. The average of Sachin after 80 one day matches is 55. If Sachin's target is increase his average to 57 after the next one day match, how much should Sachin score in the next one day match. (Average = total run / total number of innings)

   1. 213 2. 214 3. 215 4. 216 5. 217

3. When a student weighing 68 kgs joins a group of 8 students, the average of the group increases by 1.5 kgs. Find the average of the original group.

   1. 53 2. 54.5 3. 56 4. 57.5 5. None of these

4. From a group of 12 students with average weight being 72.5 kgs, two students leave. Because the two leave, the average of the group falls to 71.25 kgs. Find the average weight of the two students who leave the group.

   1. 78.5 2. 78.75 3. 79.25 4. 157 5. 157.5

5. If a student weighing 70 kgs joins a group of n students, the average of the group increases by 1 kgs. If the new student weighed 55 kgs, the average of the group would have declined by 2 kgs. Find n.

   1. 3 2. 4 3. 5 4. 6 5. 7

6. Ten years ago, the average age of a couple was 27 years old. Today also the average age of the couple and their baby is 27 years old. Find the present age of the baby.

   1. 3 2. 4 3. 5 4. 6 5. 7

7. By what does the average of all odd numbers between 50 and 130 differ from the average of all multiples of 3 between 20 and 160.

   1. 0 2. 1 3. 2 4. 3 5. 4

8. The average of n consecutive natural numbers is 17.5. What is the largest value that n can assume?

   1. Infinite 2. 34 3. 35 4. 36 5. 18

Weighted Average

The average age of 12 girls in a class is 22 years and of the 18 boys in the class is 24 years. Find the average age of the class.

Would the average age of the class be the arithmetic mean of 22 years and 24 years i.e. $\frac{22+24}{2}=23$?

We have already learnt to process the data 'average age of 12 girls is 22 years' as sum of ages of all girls = 22 × 12 = 264 and 'average age of 18 boys is 24 years' as sum of ages of all boys = 24 × 18 = 432

Thus, the sum of ages of all the 12 + 18 = 30 students of the class is 264 + 432 = 696. And the required average is $\frac{696}{30}=23.2$ years. Which is not same as the arithmetic mean of 22 and 24.

There were two 'groups' of students one with average age 22 years and another with average age 24 years. Yet the average age of the two groups together was not 23 kgs (the mean of 22 and 24). This is because the number of boys was greater than the number of girls. Thus the average of the class was closer to the average age of the boys (24 kgs) i.e. the average age of the boys had a greater influence, greater weight on the average of the entire class. Thus in this question we are finding the average of 22 years and 24 years but with respective weights (influence) of 12 and 18.

We are finding the average age i.e. the average of 22 years and 24 years. And the calculation done is $\frac{22\times12+24\times18}{12+18}$

This funda of finding the average is called Weighted Average. And as we have seen while solving the question, the formula for weighted average is:

$$C_{avg}=\frac{C_1\times w_1+C_2\times w_2}{w_1+w_2}$$

where C_1 and C_2 are the two quantities whose average is being found and w_1 and w_2 are the weights of each group.

E.g. 6: I bought 5 kgs of rice costing Rs. 32 per kg and another 7 kgs of rice costing Rs. 50 per kg. What is the average cost price per kg of rice I bought?

Rather than individually finding the amount spent on the two varieties and then dividing the total cost by the total quantity, one should be able to see this entire working in the single expression:

$$C_{avg} = \frac{\overbrace{\underbrace{32 \text{ rs/kg} \times 5 \text{ kg}}^{\text{amount spent on 1st variety}} + \underbrace{50 \text{ rs/kg} \times 7 \text{ kg}}^{\text{amount spent on 2nd variety}}}^{\text{total amount spent on the two varieties}}}{\underbrace{5+7}_{\text{total kgs bought}}} = \frac{160+350}{12} = \frac{510}{12} = 42.5$$

Beginner's Error

Many students get confused in identifying the C's and the w's incorrectly. Thus, they are not sure of will the expression be $\frac{32 \times 5 + 50 \times 7}{5+7}$ or $\frac{32 \times 5 + 50 \times 7}{32+50}$.

Thoughts that will help you correctly identify the C's and w's

1. Average of what is being found? In this case the average cost price is being found. Thus, the C's will refer to the cost price of the two groups/varieties

2. Since the unit of the answer will be Rs/kg, the numerator in the expression should refer to Rs and the denominator of the expression will refer to kgs. Thus the expression should be $\frac{\text{Total amount spent}}{\text{Total kgs bought}}$

3. Atleast in the first few questions make sure you give a meaning to each of the three brackets in the expression $\frac{\overbrace{(32 \times 5)}^{\text{amount}} + \overbrace{(50 \times 7)}^{\text{amount}}}{\underbrace{(5+7)}_{\text{total kgs}}}$

E.g. 7: Amit borrowed Rs. 6000 at rate of 4% and Rs. 4000 at rate of 6%. If the interest rates are simple interest rates, find the average rate Amit has to pay on the entire borrowed amount.

$$C_{avg} = \frac{\overbrace{\underbrace{4\% \times 6000}^{\text{interest on Rs. 6000}} + \underbrace{6\% \times 4000}^{\text{interest on Rs. 4000}}}^{\text{total interest paid on entire amount}}}{\underbrace{6000+4000}_{\text{total loan}}} = \frac{240+240}{10000} = \frac{480}{10000} = 4.8\%$$

Scenarios:

As seen by the above examples, the scenarios/context could be a class divided among boys & girls, two varieties of rice being bought or loan being broken in two lots with different interest rates. There can be many more scenarios. The following lists a few scenarios and also gives an idea about what the C's will be and what the w's will be.

1. In case of finding average profit percentage when two lots are sold at different profit percentages, the C's are the profit percentages on the lots and the w's is the amount spent as costs on the lots. Remember that $C \times w$ will be the actual profit amount when C is the profit percentage and w is the cost. Further to find the average profit percentage, the expression should be $\frac{\text{total profit}}{\text{total cost}}$ and thus, the w's are the costs incurred on the two lots.

The above is also true when we talk of interest rates as in earlier example where C's will be the interest rates on the different lots and w will be the investments made in different lots.

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2. Most common scenarios will be when two solutions, each of them being a mixture of milk and water, are mixed together. In this scenario, the C's refer to the proportion of milk (or water) in the solutions i.e. $\frac{\text{milk quantity}}{\text{total volume}}$ expressed as a ratio or as a percentage. And the w's refer to the volume of the solutions. In this case, if C is proportion of milk and w is volume of solution then $C \times w$ would refer to the quantity of milk.

3. This is an interesting case: When a journey is covered at different speeds and we need to find the average speed, obviously, since we are finding average speed, the C's would be the different speeds at which the journey is made. What is more important in this case is to identify what will the w's be.

Consider the expression 'Speed × w'. For w being what variable will this expression have some meaning? Another way to think is that to find the average speed, the final expression should be of the form $\frac{\text{total distance}}{\text{total time}}$. Thus, the denominator, $w_1 + w_2$ should refer to the total time and thus, the w_1 and w_2 should refer to the time travelled at different speeds. With w as the time, $C \times w$ would become Speed × time and would refer to the distance. Thus the numerator will become the total distance covered.

Alligation/Balancing Scale

For each of the following cases, consider the expression to find the average of C_1 and C_2, with their respective weights being

i. 18 and 30 ii. 21 and 35 iii. in the ratio 3 : 5

The expressions for the three cases are:

i. $\frac{C_1 \times 18 + C_2 \times 30}{48}$. Dividing both numerator and denominator by 6, we get

$$\frac{C_1 \times 3 + C_2 \times 5}{8}$$

ii. $\frac{C_1 \times 21 + C_2 \times 35}{56}$. Dividing both numerator and denominator by 7, we get

$$\frac{C_1 \times 3 + C_2 \times 5}{8}$$

iii. $\frac{C_1 \times 3k + C_2 \times 5k}{8k}$. Dividing both numerator and denominator by k, we get

$$\frac{C_1 \times 3 + C_2 \times 5}{8}$$

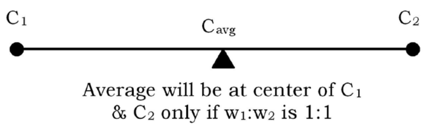
Thus, in each of the case, the average will be the same. This shows that it is not exactly the absolute values of w_1 and w_2 that determine where the average will lie between C_1 and C_2, but what is most important is the ratio $w_1 : w_2$. Thus, even if we do not the values of w_1 and w_2 but know the ratio of them, we can determine the average.

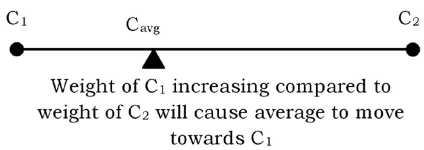
Alligation or Balancing Scale Approach – Important

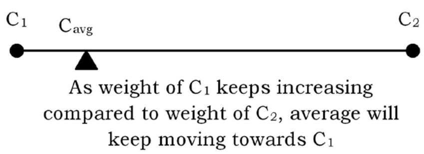
Some observations about the average of C_1 and C_2 that should be intuitive are:

1. The average will lie between C_1 and C_2 itself.
2. The average will be exactly in the middle of C_1 and C_2 only when the weights of them are equal i.e. $\frac{C_1 \times k + C_2 \times k}{2k} = \frac{C_1 + C_2}{2}$
3. If the weight of C_1 is far more than the weight of C_2, the average will move closer to C_1 and away from C_2. As the weight of C_1 keeps increasing as compared to the weight of C_2, the average also keeps becoming closer to C_1 and more away from C_2.

The above intuitive observations are captured in the following visual images:







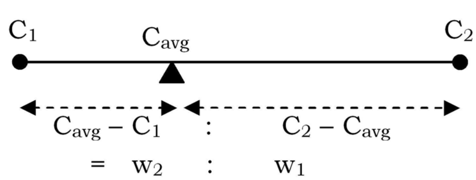
The above intuitive thoughts are now proven using mathematical expressions.

Consider re-arranging the expression of weighted averages,

$$C_{avg} = \frac{C_1 \times w_1 + C_2 \times w_2}{w_1 + w_2} \Rightarrow C_{avg} \times w_1 + C_{avg} \times w_2 = C_1 \times w_1 + C_2 \times w_2$$

i.e. $(C_{avg} - C_1) \times w_1 = (C_2 - C_{avg}) \times w_2 \Rightarrow \frac{w_1}{w_2} = \frac{C_2 - C_{avg}}{C_{avg} - C_1}$

Thus, the ratio $w_1 : w_2$ will be the (distance between C_2 & C_{avg}) : (Distance between C_{avg} & C_1). This on the visual scale will be



Please note that the weight of C_1 is the length of that part of the scale lying towards C_2 and the weight of C_2 is the length of that part of the scale lying towards C_1.

A better way to approach problems of weighted average is using the above visual alligation/scale approach. This method seems a little more demanding in thought process than using the formula, so we leave it to the students to use whichever one wants. Either of the approaches can be used for any question.

How to think while using the alligation/scale approach

While using this approach the following thought process will be very handy:

Case 1: C_1, C_2 and w_1: w_2 is given and C_{avg} is to be found out

The scale length ($C_2 - C_1$), which is known, has to be divided in the ratio $w_1 : w_2$.

Once the two parts are known, think logically towards which end will the average lie and accordingly add/subtract either of the part from C_1 or C_2

E.g. Find the average of 30 and 80 with respective weights being in the ratio 1 : 3.

Dividing the length of the scale, 50, in the ratio 1 : 3, the two parts are $\frac{1}{4}\times 50$ and $\frac{3}{4}\times 50$

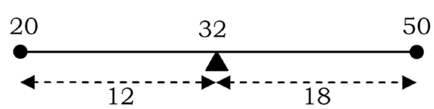
i.e. 12.5 and 37.5

Since the weight of 80 is more, the average will be closer to 80 (or away from 30) and hence the average will be 80 – 12.5 = 67.5 (or 30 + 37.5 = 67.5). You could do either of the above and thus, finding any one part is enough

Case 2: C_1, C_2 and C_{avg} is given and we need to find the $w_1 : w_2$

This is the most simple case. Just find the distances $C_{avg} - C_1$ and $C_2 - C_{avg}$ and then reverse this ratio to find $w_1 : w_2$

E.g. If the average of 20 and 50 is 32, find the ratio of the weights.



Thus, the required ratio is 18 : 12 i.e. 3 : 2

Now try the already solved examples using this technique.

Very often questions of weighted averages are camouflaged as questions involving mixing of two or more mixtures of milk and water. In such scenarios, the percentage of milk (or that of water) in the mixtures that are mixed are C_1 and C_2 and the quantity or volume of the mixtures are the w_1 and w_2.

E.g. 8: 9 litres of milk and water solution having 40% milk is mixed with 3 litres of milk and water solution having 70% milk. Find the percentage of milk in the mixture of the two solutions.

While the context here is that of two solutions being mixed, the funda is same as finding the weighted average of 40% and 70% with weights being 9 litres and 3 litres.

Method 1: Weighted Average

$$C_{avg} = \frac{\overbrace{\overbrace{40\%\times 9}^{\text{qty of milk from 1st soln}} + \overbrace{70\%\times 3}^{\text{qty of milk from 2nd soln}}}^{\text{total milk quantity after two soln are mixed}}}{\underbrace{9+3}_{\substack{\text{total volume}\\ \text{after mixing}}}} = \frac{360+210}{12} = \frac{570}{12} = 47.5\%$$

Do we have to work on milk percentage of water percentage?

You can work on either, but be consistent. If the water percentage is used for one solution, $C \times w$ will result in the quantity of water. This HAS to be added to the quantity of water (and not milk) from the other mixture. Thus, in the second solution also one has to take the water percentage. Also the answer we get will be the water percentage in the resultant mixture.

Method 2: Alligation/Scale Approach

We would just need to divide 70% – 40 % = 30% in the ratio 1 : 3. Thus the first part will be $\frac{1}{4} \times 30 = 7.5$. Since more volume of 40% milk is present required percentage has to be closer to 40% than to 70% and the required answer is 40% + 7.5 = 47.5%

The following examples are exactly similar to the example 8. It's just that among C_1, C_2, C_{avg} and w_1 & w_2, different variables are given and different variables are asked. Also the format of giving the data is different in each of the examples.

E.g. 9: When a milk and water solution having 42% milk is mixed with another solution having 57% milk, the resultant mixture has equal amount of water and milk. Find the ratio in which the two solutions are mixed.

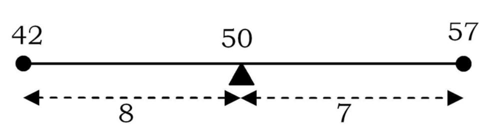
Resultant mixture having equal amount of milk and water is equivalent to it having 50% milk.

Method 1: Weighted Average

$$50 = \frac{42 \times w_1 + 57 \times w_2}{w_1 + w_2} \Rightarrow 8w_1 = 7w_2 \text{ i.e. } \frac{w_1}{w_2} = \frac{7}{8}$$

Method 2: Alligation/Scale Approach

Using the visual,



Thus the ratio of mixing is 7 : 8.

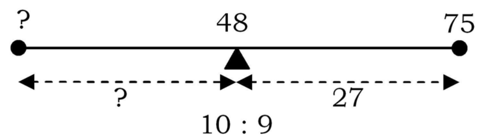
E.g. 10: When 18 litres of milk and water solution is mixed with 20 litres of milk and water solution having 75% percent of milk, the resultant mixture has 48% milk. Find the percentage of milk in the first solution.

Reducing the weights, 18 litres and 20 litres, to the reduced form, $w_1 = 9$ and $w_2 = 10$,

Method 1: Weighted Average

$$48 = \frac{C_1 \times 9 + 75 \times 10}{9 + 10} \Rightarrow 9C_1 + 750 = 912 \text{ i.e. } C_1 = 18\%$$

Method 2: Alligation/Scale Approach



Thus the line segment corresponding to the 10 part will be 30 and the percent of milk in the first solution will be 48 – 30 = 18%

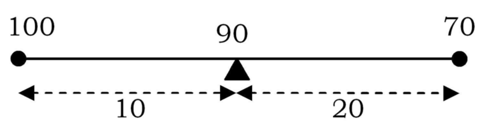
E.g. 11: How many litres of milk needs to be added to 15 litres of water and milk solution having 70% milk to result in a solution having 90% milk?

In this example while one solution is a mixture of water and milk with 70% milk, the other solution being added is pure milk i.e. 100% milk.

Method 1: Weighted Average

$$90 = \frac{100 \times w_1 + 70 \times 15}{w_1 + 15} \Rightarrow 1350 = 10w_1 + 1050 \text{ i.e. } w_1 = 30$$

Method 2: Alligation/Scale Approach



Thus milk and 70% milk solution has to be mixed in the ratio 2 : 1. Since we have 15 litres of 70% milk solution, we need to add 30 litres of milk to it.

E.g. 12: Two solutions having milk and water in the ratios 3 : 5 and 4 : 7 are mixed in the ratio 2 : 3. Find the ratio of milk and water in the resultant mixture.

Also please note that 3 : 5 and 4 : 7 are the ratio of milk and water in the two solutions respectively, whereas 2 : 3 is the ratio of the quantities of first solution and second solution. Thus C_1 and C_2 will be derived using 3 : 5 and

4 : 7 and the weights $w_1 : w_2$ will be 2 : 3. Quite a few beginners get puzzled when so many ratios appear in the same sentence. Just ask yourself which ratio talks about volumes to the two solutions to identify which results in w_1 and w_2.

In this example, instead of giving the percentage of milk directly, the ratio of milk and water is given. The percent of milk in the first solution is NOT $\frac{3}{5}$ but is $\frac{3}{8}$ i.e. 37.5%. Similarly the percentage of milk in second is $\frac{4}{11}$ i.e. 36.36%. Since the percentages are not very manageable we will use itself $\frac{3}{8}$ & $\frac{4}{11}$.

Method 1: Weighted Average

$$C_{avg} = \frac{\frac{3}{8}\times 2 + \frac{4}{11}\times 3}{2+3} = \frac{\frac{3}{4}+\frac{12}{11}}{5} = \frac{33+48}{220} = \frac{81}{220}$$

This, $\frac{81}{220}$ is not the ratio of milk and water in the resultant but is the proportion of milk. This ratio implies that if the total volume is 220 litres, milk will be 81 litres. Thus, ratio of milk and water in the resultant is 81 : (220 – 81) i.e. 81 : 139.

Why not find weighted average of $\frac{\text{milk}}{\text{water}}$ itself?

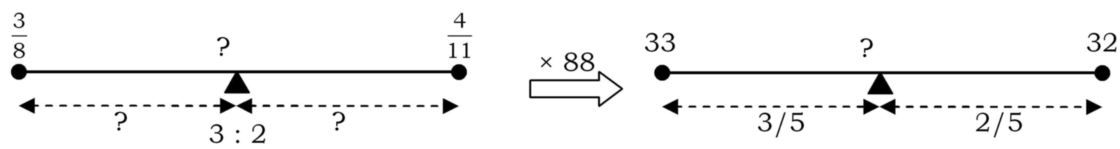
Since in the question the data given is that of milk : water and the question asked is also about milk : water, why do we go the round-about way of changing all data to $\frac{\text{milk}}{\text{total}}$ and in the end change the found value of $\frac{\text{milk}}{\text{total}}$ back to milk : water?

Why not work directly in $\frac{\text{milk}}{\text{water}}$ as $C_{avg} = \frac{\frac{3}{5}\times 2 + \frac{4}{7}\times 3}{2+3}$?

Because the term $\frac{3}{5}^{th}\times 2$ litres or $\frac{4}{7}^{th}\times 3$ litres in the numerator does not mean anything. $\frac{3}{8}^{th}$ & $\frac{4}{11}^{th}$ have the meaning because they refer to the fraction of the total solution that is milk and hence $\frac{3}{8}^{th}\times 2$ litres is the milk that is obtained form first solution and $\frac{4}{11}^{th}\times 3$ litres is the milk that is obtained from the second solution. But no such meaning can be assigned to $\frac{3}{5}^{th}\times 2$ litres or $\frac{4}{7}^{th}\times 3$ litres.

Method 2: Alligation/Scale Approach

Using the visual means in this case would also require smart work to avoid calculation



Thus, the average, after being multiplied with 88 will be

$33-\frac{3}{5}$ or $32+\frac{2}{5}$ i.e. $\frac{162}{5}$. (Notice that the scale here is decreasing from left to right and hence going rightwards, one has to subtract. Anyways the average should be between 33 and 32.)

Thus, the required average ratio of milk to total is $\frac{162}{5\times 88}=\frac{81}{220}$

Thus ratio of milk and water is 81 : 139

Average when more than two groups/varieties are mixed.

When more than two groups are mixed, working the visual way (alligation) is going to be very very cumbersome (can only work by treating them a sequence of mixing two at a time). In this situation, its best to use the formula of weighted average. For more than two groups the weighted average formula can be expanded as follows:

$$C_{avg}=\frac{C_1\times w_1+C_2\times w_2+C_3\times w_3+\ldots\ldots}{w_1+w_2+w_3}$$

Exercise

9. Two alloys of iron having 85% and 67% iron respectively are fused together to form another alloy having 71% iron in it. If 14 kgs of the first alloy is taken, find the weight of the second alloy.

1. 28 kgs 2. 34 kgs 3. 35 kgs 4. 42 kgs 5. 49 kgs

10. A trader sells rice to customers at a profit of 20%. But to his regular customers he charges a profit of only 10%. If he makes an overall profit of 18.18...%, what fraction of his customers are regular customers?

1. $2/9$ 2. $9/11$ 3. $2/11$ 4. $2/7$ 5. $7/9$

11. In my written test I scored 75% marks, but because of the viva, my overall percentage decreased to 68%. If the written test was conducted for 500 marks and the viva for 100 marks, find the percentage of marks scored in the viva.

1. 15% 2. 16.66% 3. 25% 4. 33% 5. 40%

12. How many litres of solutions having 30% milk and 70% milk should be mixed so that we get 40 litres of 45% milk solution?

1. 10 , 30 2. 12 , 28 3. 15 , 25 4. 20 , 20 5. 25 , 15

13. Two vessels having volumes in the ratio 3 : 5 are filled with water and milk solutions. The ratio of milk and water in the two vessels is 2 : 3 and 3 : 1 respectively. If the contents of both the vessel are emptied into a larger vessel, find the ratio of milk and water in the larger vessel.

1. 99 : 61 2. 99 : 160 3. 61 : 160 4. 61 : 99 5. None of these

14. 12 litres of a milk and water solution having ratio of milk and water as 4 : 1 is mixed with 20 litres of another milk and water solution having ratio of milk and water in the ratio 5 : 1. Find the ratio of milk and water in the resultant solution.

1. 43 : 240 2. 43 : 197 3. 197 : 240 4. 197 : 43 5. None of these

15. A milk-man mixes water equal to 20% of the quantity of milk he has, to the milk. Suspecting that his customers will notice the dilution, he decides to add pure milk to the mixture such that the ratio of milk and water increases to 17 : 3. If he had added 12 litres of water, how much litres of milk will he need to add.

    1. 6 2. 8 3. 10 4. 12 5. 15

16. Contents of three vessels having volumes in the ratio of 2 : 3 : 4 and having solutions with milk concentrations of 40%, 30% and 20% are mixed together in a fourth large vessel. Find the concentration of milk after the mixing.

    1. 30 % 2. 27.7 % 3. 27.2 % 4. 26.1 % 5. 33 %

17. Ramesh sells one-third of his goods at a profit of 20 %. One-third of the remaining goods get damaged and have to sold at a loss of 10 %. At what minimum profit percentage should the rest of the goods be sold such that Ramesh makes an overall profit of atleast 10%?

    1. 16.25 % 2. 15 % 3. 14.28 % 4. 12.5 % 5. 10 %

18. A solution of milk and water having 20% milk is mixed with another solution having 50% milk in the ratio 2 : 5. If 21 lts of this mixture is taken and mixed with 9 lts of another milk and water solution having 80% milk, find the percentage of milk in the final mixture.

    1. 43 % 2. 47 % 3. 50 % 4. 53 % 5. 57 %

19. How many litres of water needs to be added to 25 litres of a solution having milk and water in the ratio 8 : 5, such that the resultant has milk and water in the ratio 5 : 8.

    1. 15 2. 20 3. 25 4. 30 5. 4

20. 12 litres of milk is added to 30 litres of solution having milk and water in the ratio 3 : 5. Find the ratio of milk and water after the addition.

    1. 22 : 25 2. 34 : 25 3. 9 : 25 4. 56 : 25 5. 31 : 25

Adding a Pure Solution to a Mixture

In the last two questions (Q. no. 19 & 20) of the previous exercise, a pure solution was added to a mixture. In Q. 19, water was being added to a mixture of milk and water and in Q. 20, milk was being added to a mixture of milk and water. These types of questions can be solved using the funda of weighted average by taking percentage of milk as 100% in milk and as 0% in water. There exists a more efficient way of solving such questions – using variation. This method is important not only because it saves time but also because further ahead we would have three or more instances of adding pure component to a mixture in the same question. And using weighted average there would be cumbersome.

When a pure component, say milk, is added to a solution of milk and water, the percentage of water (that part who's quantity does not change because of addition) varies inversely with the volume of the solution i.e. $C \alpha \frac{1}{V}$. Thus $C \times V$ = constant and if volume changes from V_i to V_f and the respective concentrations were C_i and C_f, we would have $C_f \times V_f = C_i \times V_i$. (The subscripts used are f for final and i for initial)

Also this is obvious because, if milk is added to a solution of milk and water, working on quantity of water, $C_i \times V_i$ refers to the amount of water before milk is added and $C_f \times V_f$ refers to the amount of water after milk is added. And the two amounts have to be the same because no water is added.

Solving the last two questions of the previous exercise using this approach:

Q. 19: How many litres of water needs to be added to 25 litres of a solution having milk and water in the ratio 8 : 5, such that the resultant has milk and water in the ratio 5 : 8.

Since water is being added, working on quantity of milk, $\frac{8}{13} \times 25 = \frac{5}{13} \times V_f \Rightarrow$

V_f = 40 litres. Thus, water added would be 40 – 25 = 15 litres.

Q. 20: 12 litres of milk is added to 30 litres of solution having milk and water in the ratio 3 : 5. Find the ratio of milk and water after the addition.

Since milk is being added, working on quantity of water, $\frac{5}{8} \times 30 = C_f \times 42 \Rightarrow C_f = \frac{25}{56}$

Thus the final proportion of water to the total solution would be $\frac{25}{56}$ and thus the required ratio of milk and water is (56 – 25) : 25 i.e. 31 : 25.

Removal and Replacement

A standard type of question where the above will be useful is:

10 litres are removed from 80 litres of solution of milk and water having ratio of milk and water as 4 : 3 and replaced with water. This operation is done a total of three times. Find the ratio of milk and water in the solution now. Also find the quantity of milk in the solution.

The removal and replacement should be considered as two operations done in succession …

i. Solution is removed

ii. Water is being added to the solution.

It is important to think of the operation as the above two sub-operations. One of the operation will not have any impact on the answer and the other will be significant in finding the answer.

The above type of question can be solved by two seemingly similar but logically very different approaches. At the risk of being confusing, we are providing both the thought process because based on what the question is asking, one approach is preferable to the other.

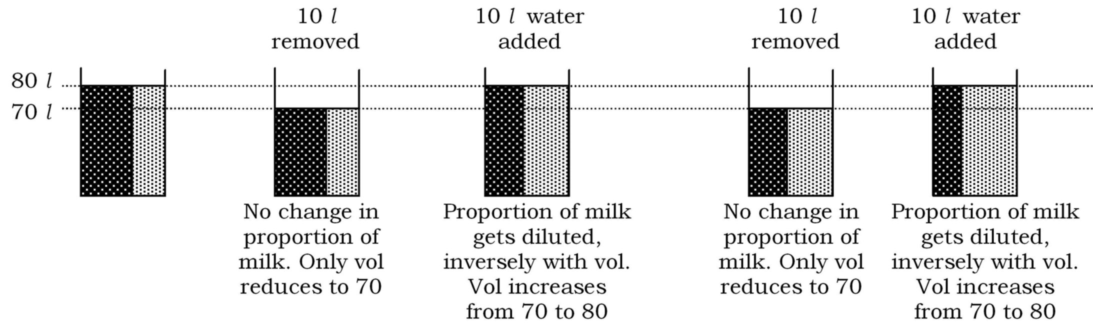
Approach 1: Working on **proportion**

i. Solution is removed: While removing a solution, since it is a homogenous solution, the ratio of milk and water in the residual solution is same as that in the original solution. This should be understood without the use of any maths. Say a milk man is delivering milk that is diluted with water. After delivering milk to a household, the milk that is left in his can will have milk and water in the same ratio as that before delivering.

ii. Water is being added to the solution: While replacing with water, the proportion of milk in the solution gets reduced. And in this step, we can use what was learnt above since a pure solution, water, is being added. Thus, the proportion of milk varies inversely with the volume.

Now the solution can be understood better through the following visual:

Working on Proportion



<u>**Proportion**</u> of milk in soln: $\frac{4}{7}$ No change, $\frac{4}{7}$ $\frac{4}{7}\times\frac{7}{8}=\frac{1}{2}$ No change, $\frac{1}{2}$ $\frac{1}{2}\times\frac{7}{8}=\frac{7}{16}$

While the above seems quite a bit of working, it is given visually so that you can realize that in the three operations, the proportion of milk gets diluted thrice by a factor of $\frac{7}{8}$. Thus, the final proportion of milk will be $\frac{4}{7}\times\frac{7}{8}\times\frac{7}{8}\times\frac{7}{8}=\frac{49}{128}$.

Thus, the ratio of milk and water in the final solution is 49 : (128 – 49) i.e. 49 : 79.

And since the total volume of the solution is 80 litres, the quantity of milk in the solution will be $\frac{49}{128}\times 80=\frac{245}{8}$ litres.

Factors to be kept in mind while working on <u>proportion</u>

1. First check which pure part is being added to the solution. If water is added, we will work on proportion of milk. And if milk is being added we HAVE to work on proportion of water. For the next points, we will assume water is being added.

2. The use of solution being removed is just to find the volume after removal, say, V_i

3. When water is added, the volume increases to V_f. It is only in this step that the proportion of milk is diluted and becomes $C_i\times\frac{V_i}{V_f}$. Since this is a case of dilution, the factor $\frac{V_i}{V_f}$ has to be less than 1.

Occasionally the volume removed and that added will be different. Also occasionally water may be added first and then solution removed.

The focus should be just on the operation of adding water. And we should ask ourselves what was the volume before adding and what is the volume after adding.

4. Step 3 has to be repeated for every addition of water.

5. In the end we would be arriving at the proportion of milk in the solution. If the quantity of milk is asked, it can be found by multiplying the proportion of milk with the total volume.

Approach 2: Working on **quantity**

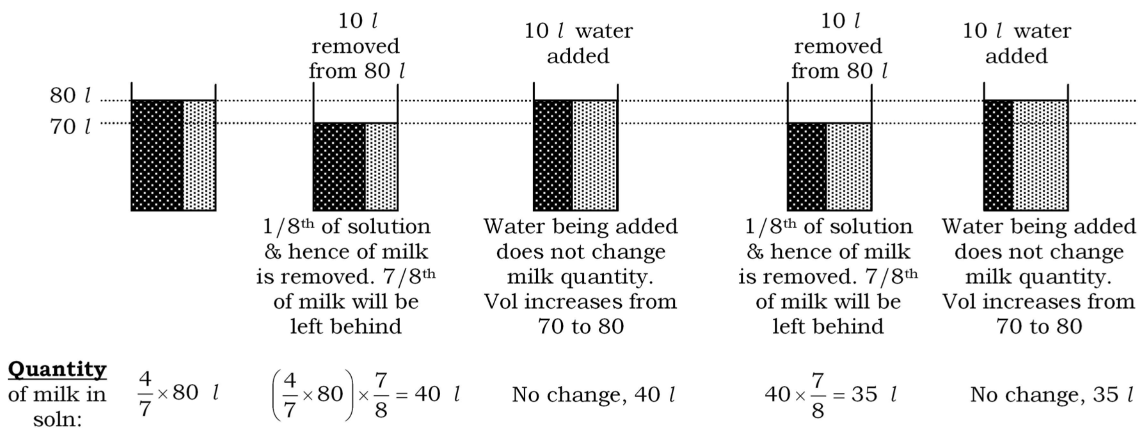
i. Solution is removed: While removing a solution, since it is a homogenous solution, both milk and water is going to be removed. Thus, the quantity of milk gets reduced in this operation.

Further if the fraction of the solution being removed is *f*, then the fraction of milk that will be removed is also *f*. This should be understood without the use of any maths. Say a milk man is delivering milk that is diluted with water. If to a household he delivers half his can, then half of the milk present in the can will be delivered (along with half of the water); if he delivers $1/4^{th}$ of the can to the household, then $1/4^{th}$ of the milk will be delivered (along with $1/4^{th}$ of the water).

But we are interested in the volume left behind and not in that which is being removed. The fraction left behind will be $(1 - f)$.

ii. Water is being added to the solution: While replacing with water, the quantity of milk in the solution will not change. Thus, this operation can be ignored to a large extent and one just needs to get an idea of what the final volume is.

Working on Quantity



Thus, the final quantity of milk can be found as $\left(\frac{4}{7} \times 80\right) \times \frac{7}{8} \times \frac{7}{8} \times \frac{7}{8} = \frac{245}{8}$ litres.

Since the final volume of the solution will be 80 lts, water quantity can be found as

$80 - \frac{245}{8} = \frac{395}{8}$

And the required ratio will be 245 : 395 i.e. 49 : 79.

Factors to be kept in mind while working on Quantity

1. First check which pure part is being added to the solution. If water is added, we will work on quantity of milk. And if milk is being added we HAVE to work on quantity of water. For the next points, we will assume water is being added.

2. When solution is being removed, quantity of milk will also reduce.

The focus should be just on the operation of removal of solution. And we should ask ourselves what fraction of the volume is being removed. If *f* is the fraction being removed, quickly move your focus to fraction left behind i.e. (1 – *f*).

3. Water being added does not change the quantity of milk and can be ignored. At best it is important to find the volume from which the next removal will happen, to find the fraction being removed.

4. Step 2 has to be repeated for every time that solution is removed.

5. In the end we would be arriving at the quantity of milk in the solution. If the ratio of milk and water is asked, the amount of water can be found by subtracting the amount of milk from total volume. And then the ratio can be taken.

E.g. 13: A solution has 60% milk and rest water. If 10% of it is removed and replaced with water thrice, what is the percentage of milk left in the solution?

Approach 1: Working on proportion of milk

When water is being added, each of the three times, the volume increases from 0.9V to V. Thus the proportion of milk will get diluted by a factor of $\frac{9}{10}$

Thus, final proportion of milk = $60\% \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}$ = 31.74%

Approach 2: Working on quantity of milk

On each of the three occasion when solution is being removed, 10% is being removed i.e. 90% of the solution is left behind. Thus, on each of the three occasion, 90% of the milk present will be left behind i.e. final quantity of milk = $(60\% \times V) \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}$. Finally the volume of the entire solution will be back to V and thus, the proportion of milk will be

$$\frac{(60\% \times \text{V}) \times \frac{9}{10} \times \frac{9}{10} \times \frac{9}{10}}{\text{V}} = 31.74\%$$

E.g. 14: From a can containing milk, 10% is removed and replaced with water. Next 11.11% of the solution is removed and replaced with water. Further 12.5% of the solution is removed and replaced with water. What is the percentage of milk in the solution now?

Approach 2: Working with quantity of milk.

In the first instance of removal, 10% of milk is removed i.e. 90% of milk is left behind.

In the second instance of removal, 11.11% i.e. $\frac{1}{9}^{th}$ of milk is removed i.e. $\frac{8}{9}^{th}$ of milk is left behind.

In the third instance of removal, 12.5% i.e. $\frac{1}{8}^{th}$ of milk is removed i.e. $\frac{7}{8}^{th}$ of milk is left behind.

Thus, if V was the initial volume, the final quantity of milk left behind will be $V \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8} = 0.7V$.

Since each time the replaced quantity is exactly equal to the removed quantity, the final volume will also be V and thus the percentage of milk will be $\frac{0.7V}{V} \times 100 = 70\%$

Approach 1: Working on proportion of milk.

In the first instance of water being added, the volume will increase from 0.9V to V i.e. the proportion of milk will dilute by the factor $\frac{9}{10}$

In the second instance, after removal only $\frac{8}{9}V$ is present which will increase to V when water is added. Thus, the proportion of milk will dilute by the factor $\frac{8}{9}$

In the second instance, after removal only $\frac{7}{8}V$ is present which will increase to V when water is added. Thus, the proportion of milk will dilute by the factor $\frac{7}{9}$

Since initially the proportion of milk was 100%, the final proportion of milk is $100 \times \frac{9}{10} \times \frac{8}{9} \times \frac{7}{8} = 70\%$.

E.g. 15: From a 100 ml can full of milk, 10 ml is removed and 20 ml water is added. Next 20 ml of the solution is removed and 30 ml water is added. What is the quantity of milk in the solution now?

Approach 1: Working on proportion of milk

When 10 ml of milk is removed, 90 ml of milk is left. And this gets increased to 110 ml, when water is added. Thus, proportion of milk gets diluted by the factor $\frac{9}{11}$

Next when 20 ml of solution is removed, 90 ml of solution is left behind. And this gets increased to 120 ml when water is added. Thus, proportion of milk gets diluted by the factor $\frac{9}{12}$ i.e. $\frac{3}{4}$

Thus, final proportion of milk = $1 \times \frac{9}{11} \times \frac{3}{4} = \frac{27}{44}$ and the quantity of milk will be $\frac{27}{44} \times 120 \text{ lts} = \frac{810}{11}$ lts.

Approach 2: Working on quantity of milk

In the first instance of removal, 10 lts is removed from 100 lts i.e. $\frac{1}{10}^{th}$ is removed i.e. $\frac{9}{10}^{th}$ of milk will be left behind.

In the second instance of removal, 20 lts is removed from 110 lts i.e. $\frac{2}{11}^{th}$ is removed i.e. $\frac{9}{11}^{th}$ of milk will be left behind.

Thus, final quantity of milk = $100 \text{ lts} \times \frac{9}{10} \times \frac{9}{11} = \frac{810}{11}$ lts

E.g. 16: From a solution containing milk and water in the ratio 2 : 3, 15 litres of solution is removed and replaced with 15 litres of milk. This operation is done once more. Now the ratio of milk to water is 7 : 5. Find the volume of the initial solution.

In this example, we will have to work on percentage of water as milk is being added back.

Approach 1: Working on proportion of water:

There are two instances when milk is being added. If V_i and V_f are the volume just before adding the milk and just after adding the milk, these would be the same for both the instances of milk being added. Thus, in both the instances the proportion of milk is being diluted by the same factor, $\frac{V_i}{V_f}$

Thus, $\frac{5}{12} = \frac{3}{5} \times \left(\frac{V_i}{V_f}\right)^2 \Rightarrow \frac{V_i}{V_f} = \frac{5}{6}$

We also know that the difference between V_i and V_f is 15 litres since 15 litres of milk is added. Hence searching for two numbers in ratio 5 : 6 with difference 15, we can find V_i = 75 litres and V_f = 90 litres.

Thus, originally 90 litres of solution was present which after removal became 75 lts and after adding milk again became 90 lts.

Approach 2: Working on quantity of water.

Since initial and final volume of entire solution is same, say V, then

Final Qty of water = Initial Qty of water × f^2, where f is the fraction of milk left behind after solution is removed.

Thus, $\frac{5}{12}V = \frac{3}{5}V \times f^2 \Rightarrow f = \frac{5}{6}$

Since solution left behind after removal is $\frac{5}{6}^{th}$, then $\frac{1}{6}^{th}$ of the solution must have been removed. And since this is given to be 15 lts, the total volume would have been 15 × 6 = 90 lts.

E.g.17: From 100 litres of milk, 10 litres of water is added and then 20 litres of solution is removed. Next 30 litres of the water is added and 20 litres of solution is removed. Find the amount of milk, in litres, in the solution now.

Approach 1: Working on proportion of milk

Remember that we need to just focus on the operation of addition of water and ask ourselves what was the volume before addition and after addition.

In the first instance of addition, the volume increases from 100 lts to 110 litres. Thus the proportion of milk will get diluted by the factor $\frac{10}{11}$

In the second instance of addition, the volume increases from 90 lts to 120 litres. Thus the proportion of milk will get diluted by the factor $\frac{9}{12}$ i.e. $\frac{3}{4}$

Thus, final proportion of milk = $1 \times \frac{10}{11} \times \frac{3}{4} = \frac{15}{22}$. And since the final volume of entire solution is 100 litres, the quantity of milk will be $\frac{15}{22} \times 100 = \frac{750}{11}$ lts.

Approach 2: Working on quantity of milk

Remember that in this approach we just need to focus on the removal and ask ourselves what fraction of solution is removed.

In the first instance of removal, 20 lts is removed from 110 lts i.e. $\frac{2}{11}^{th}$ is removed i.e. $\frac{9}{11}^{th}$ of milk will be left behind.

In the second instance of removal, 20 lts is removed from 120 lts i.e. $\frac{2}{12}^{th}$ i.e. $\frac{1}{6}^{th}$ is removed i.e. $\frac{5}{6}^{th}$ of milk will be left behind.

Thus, final quantity of milk = $100 \times \frac{9}{11} \times \frac{5}{6} = \frac{750}{11}$ lts

Exercise

21. 12.5% of a solution having milk and water in the ratio 24 : 25 is removed and replaced with water. Find the ratio of milk and water in the solution if this operation is done a total of three times.

    1. 21 : 64 2. 27 : 64 3. 8 : 27 4. 27 : 33 5. 21 : 43

22. Rahul removes a fraction of undiluted spirit and replaces it with water to avoid detection. He does so on two continuous days. If spirit now accounts for only 64% of the solution, what fraction of solution did he remove daily?

    1. $\frac{1}{3}$ 2. $\frac{2}{3}$ 3. $\frac{1}{4}$ 4. $\frac{1}{5}$ 5. $\frac{1}{6}$

23. A cask is full of wine but it has a leak in the bottom. When one-fourth of the cask empties out because of the leak, the cask is replenished with water. Next when half of the cask has leaked out, it is again filled with water. Finally when three-fourths of the cask leaks out, it is again filled with water. What is the percentage of wine in the cask now?

    1. 9.375% 2. 8.33% 3. 7.2% 4. 7.5% 5. 6.66%

24. When 10% of a solution of milk and water is removed and replaced with water, the ratio of milk and water becomes 3 : 1. Find the ratio of milk and water before the water is added to the solution.

    1. 2 : 1 2. 4 : 1 3. 5 : 1 4. 5 : 2 5. 4 : 3

25. From a 100 litre container having milk and water in the ratio 8 : 7, 10 litres of solution is removed and replaced with 20 litres of milk. Again 20 litres of solution is removed and replaced with 30 litres of milk. Find the amount of milk in the solution now.

    1. 112.5 2. 105.6 3. 93.7 4. 87.5 5. 85.6

26. 12 litres of water is added to 60 litres of milk and then 12 litres of the solution is removed. Next, 12 litres of water is added to the solution and again 12 litres of the solution is removed. Find the percentage of milk in the solution now.

    1. 75% 2. 69.44% 3. 64.44% 4. 66.66% 5. 62.5%

27. Water equal to 10% of a solution of milk and water is added to the solution. Now 10% of the solution is removed. If now the ratio of milk and water is 3 : 1, find the ratio of milk and water in the initial solution.

    1. 27 : 40 2. 27 : 13 3. 33 : 7 4. 33 : 40 5. 27 : 33

28. In a particular dilution technique, 10% of the solution is removed and replaced with the diluter. If we start with pure alcohol, minimum how many times would the operation need to be performed to bring the percentage of alcohol below 65%.

    1. 3 2. 4 3. 5 4. 6 5. 7

CAT Questions

1. *[CAT 2007]* Consider the set S = {2, 3, 4, ..., 2n+1), where n is a positive integer larger than 2007. Define X as the average of the odd integers in S and Y as the average of the even integers in S. What is the value of X – Y?

   (1) 1 (2) $\frac{n}{2}$ (3) $\frac{n+1}{2n}$ (4) 2008 (5) 0

2. *[CAT 2007]* Ten years ago, the ages of the members of a joint family of eight people added up to 231 years. Three years later, one member died at the age of 60 years and a child was born during the same year. After another three years, one more member died, again at 60, and a child was born during the same year. The current average age of this eight-member joint is nearest to

   (1) 22 years (2) 21 years (3) 25 years (4) 24 years (5) 23 years

3. *[CAT 2007]* The question is followed by two statements, A and B. Mark your answer as

   (1) if the question can be answered by using A alone but not by using B alone.
   (2) if the questions can be answer by using B alone but not by using A alone.
   (3) if the question can be answered by using both statements together but not by either statements alone.
   (4) if the question cannot be answered on the basis of the two statements.

   The average weight of a class of 100 students is 45 kg. The class consists of two sections, I and II, each with 50 students. The average weight, W_I, of section I is smaller than the average weight, W_{II}, of section II. If the heaviest student, say Deepak, of section II is moved to section I, and the lightest student, say Poonam, of section I is moved to section II, then the average weights of the two sections are switched, i.e. the average weight of Sections I becomes W_{II} and that of Section II becomes W_I. What is the weight of Poonam?

   A: $W_{II} – W_I = 1.0$
   B: Moving Deepak from Section II to I (without any move from I to II) makes the average weights of the two sections equal.

4. *[CAT 2004]* A milkman mixes 20 litres of water with 80 litres of milk. After selling one-fourth of this mixture, he adds water to replenish the quantity that he has sold. What is the current proportion of water to milk?

   (1) 2 : 3 (2) 1 : 2 (3) 1 : 3 (4) 3 : 4

5. *[CAT 2001]* A student took five papers in an examination, where the full marks were the same for each paper. His marks in these papers were in the proportion of 6 : 7 : 8 : 9 : 10. In all papers together, the candidate obtained 60% of the total marks. Then the number of papers in which he got more than 50% marks is:

   (1) 2 (2) 3 (3) 4 (4) 5

6. *[CAT 2001]* Three math classes: X. Y, and Z, take an algebra test.

   The average score in class X is 83.
   The average score in class Y is 76.
   The average score in class Z is 85.
   The average score of all students in classes X and Y together is 79.
   The average score of all students in classes Y and Z together is 81.
   What is the average for all the three classes?

   (1) 81 (2) 81.5 (3) 82 (4) 84.5

7. *[CAT 2001]* A set of consecutive positive integers beginning with 1 is written on the blackboard. A student came along and erased one number. The average of the remaining numbers is $35\frac{7}{17}$. What was the number erased?

   (1) 7 (2) 8 (3) 9 (4) None of these

8. *[CAT 2000]* Consider a sequence of seven consecutive integers. The average of the first five integers is n. The average of all the seven integers is

   (1) n (2) $n + 1$ (3) $k \times n$, where k is a function of n (4) $n + (2/7)$

Directions for 9 & 10: *[CAT 1999]* The following table presents the sweetness of different forms relative to sucrose, whose sweetness is taken to be 1.00.

Lactose	0.16
Maltose	0.32
Glucose	0.74
Sucrose	1.00
Fructose	1.70
Saccharin	675.00

9. What is the minimum amount of sucrose (to the nearest gram) that must be added to one-gram of saccharin to make a mixture that will be at least 100 times as sweet as glucose?

   (1) 7 (2) 8 (3) 9 (4) 100

10. Approximately how many times sweeter than sucrose is a mixture consisting of glucose, sucrose and fructose in the ratio of 1 : 2 : 3?

   (1) 1.3 (2) 1 (3) 0.6 (4) 2.3

Time Speed Distance

Speed is defined as distance covered per unit time. Thus,

$$\text{Speed } (S) = \frac{\text{Distance } (D)}{\text{Time } (T)}$$

One should be very conversant with all the possible arrangements of the three terms:

$D = S \times T$

$$T = \frac{D}{S}$$

Both of these formats will be used equally frequently as the first relation. So be very swift with these.

Looking at the expression of speed, the unit of speed can be deduced as $\frac{\text{km}}{\text{hr}}$ (represented in this text as kmph) or $\frac{\text{m}}{\text{s}}$ (represented in this text as m/s). In questions one would require to convert the units and hence it's good idea to memorise the conversions.

1 kmph = $\frac{1000}{3600}$ m/s i.e. $\frac{5}{18}$ m/s.

Thus, $\text{kmph} \xrightarrow{\times \frac{5}{18}} \text{m/s}$ and $\text{m/s} \xrightarrow{\times \frac{18}{5}} \text{kmph}$

> **Conversion from kmph to m/s**
>
> More often than not, in questions that require conversion of speed from kmph to m/s, the speed used will be some comfortable multiple of 18 kmph. One should not spend time in using the above conversion process, instead should be ready with the following:
>
> 18 kmph = 5 m/s   36 kmph = 10 m/s   45 kmph = 12.5 m/s
>
> 54 kmph = 15 m/s   72 kmph = 20 m/s   90 kmph = 25 m/s

The following examples and exercise is just a sort of primer to make you acquainted with using the relations learnt above. As we will learn ahead, they can be solved, using proportionality. As of now stick to learning how to form equations using the above relations.

E.g. 1: I travel three distances in the ratio 1 : 2 : 3 at speeds of 20 m/s, 30 m/s and 40 m/s. If the total time taken is 23 minutes, find the total distance covered.

In this case there are three distances travelled at three different speeds and hence there are three different time taken for the three stretches. And we are given the sum of the time take i.e.

$\text{Time}_1 + \text{Time}_2 + \text{Time}_3 = 23 \times 60$ secs.

And we know that time is found as $\frac{\text{Distance}}{\text{Speed}}$.

Thus, $\frac{x}{20} + \frac{2x}{30} + \frac{3x}{40} = 23 \times 60$

Multiplying by 120, $6x + 8x + 9x = 23 \times 60$ i.e. $x = 60$.

Thus, the total distance covered = $x + 2x + 3x = 6x = 360$ meters.

E.g. 2: I travel for three stretches of time, in the ratio 1 : 2 : 3, at speeds of 20 kmph, 30 kmph and 40 kmph. If cover a total distance of 100 km, find the total time for which I am travelling.

In this case there are three time intervals travelled at three different speeds and hence there are three different distance covered in the three stretches. And we are given the sum of these distances i.e.

$\text{Distance}_1 + \text{Distance}_2 + \text{Distance}_3 = 100$ kms.

And we know that distance is given by $D = S \times T$

Thus, $20 \times x + 30 \times 2x + 40 \times 3x = 100$ i.e. $200x = 100$ i.e. $x = ½$

We need total time i.e. $x + 2x + 3x = 6x$ and will be $6 \times ½ = 3$ hours.

Forming equations

Very often the journey would be broken into two or three parts. You have to be very agile in forming equations. You need quickly ascertain whether the equation can be formed in terms of time ($\text{Time}_1 + \text{Time}_2 + \text{Time}_3 + \ldots\ldots = \text{Total time}$) or whether the equation can be formed in terms of distance ($\text{Distance}_1 + \text{Distance}_2 + \text{Distance}_3 + \ldots\ldots = \text{Total distance}$).

Next step is to be very conversant that Time is given as $\frac{\text{Distance}}{\text{Speed}}$ whereas Distance is given by Speed × Time.

Hence look out for relations of the type

$$\frac{d_1}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + \ldots\ldots = T \text{ or } (s_1 \times t_1) + (s_2 \times t_2) + (s_3 \times t_3) + \ldots\ldots = D$$

Sometimes you can have the liberty to frame the equation in whichever term you wish, as the following example shows.

E.g. 3: I travel from Pune to Mumbai via Lonavala. The total journey of 160 km takes me 2 hours. If I travel between Pune and Lonavala at speed of 60 kmph and between Lonavala and Mumbai at speed of 90 kmph, find the distance between Lonavala and Mumbai and also the time taken for this stretch.

Approach 1: Forming equation in terms of distance

If the time taken between Pune and Lonavala is assumed as t, then the time taken between Lonavala and Mumbai will be $(2 - t)$. Thus, adding the distances covered in the two stretches,

$60 \times t + 90(2 - t) = 160$ i.e. $30t = 20$ i.e. $t = 2/3$ i.e. 40 minutes. Thus, the time taken between Lonavala and Mumbai will be 120 min – 40 min = 80 min and required distance will be $90 \times \frac{80}{60} = 120$ km.

Approach 2: Forming equation in terms of time

If the distance between Pune and Lonavala is assumed as d, then the distance between Lonavala and Mumbai is $(160 - d)$. Next, adding the time taken in the two stretches,

$\frac{d}{60} + \frac{160 - d}{90} = 2$ i.e. $3d + 2(160 - d) = 2 \times 180$ i.e. $d = 40$

Thus, the distance between Lonavala and Mumbai = 160 – 40 = 120 and the time taken = $\frac{120}{90}$ i.e. $\frac{4}{3}$ hr i.e. 80 mins.

Exercise

1. A bus travels at a speed of 45 kmph. But it stops every 250 meters at a bus-stop for 1 minute. In how many minutes does the bus cover a distance of 6 kms.

   1. 8 2. 30 3. 31 4. 32 5. None of these

2. In a relay race of 4 × 100 meters (4 sprinters each running 100 meters successively), the speed of the four athletes is 8 m/s, 6 m/s, 8 m/s and 12 m/s. Find the time in which the team completes the 400 meter race.

   1. 40 sec 2. 45 sec 3. 50 sec 4. 55 sec 5. 60 sec

3. If the difference in the time taken to cover a certain distance at 6 kmph and at 10 kmph is 30 minutes, find the distance.

   1. 6 km 2. 7.5 km 3. 8 km 4. 10 km 5. 12 km

4. While going to office, I am in a hurry and hence I travel at 60 kmph but while coming back from office I drive at a more leisurely speed of 40 kmph. If I take a total of 30 minutes for both the way, find the distance to my office.

   1. 6 km 2. 7.5 km 3. 8 km 4. 10 km 5. 12 km

5. I cover a total of 300 km in 8 hours, partly at a speed of 30 kmph and partly at a speed of 50 kmph. For what time was I traveling at a speed of 30 kmph?

   1. 2 hrs 2. 3 hrs 3. 4 hrs 4. 5 hrs 5. 6 hrs

6. Find the ratio of the time taken to cover two distances in the ratio 4 : 5 at speeds in the ratio 4 : 3 respectively.

   1. 1 : 1 2. 3 : 5 3. 5 : 3 4. 4 : 3 5. 4 : 5

7. Everyday I cover the distance from my home and office at a usual speed and take a certain time at the usual speed. When I increase my usual speed by 5 kmph, I take 10 minutes less than usual. If I reduce my usual speed by 5 kmph, I take 15 minutes more than usual. Find the distance from home to office.

   1. 25 km 2. 30 km 3. 40 km 4. 45 km 5. 50 km

Proportionality b/w Time, Speed and Distance

This is probably the most important topic in this chapter. And also the most thought-intensive. It provides the foundation to solve many tough problems orally!

Usually in questions there are two scenarios. And in the two scenarios, one among the three variables, S, D and T, is a constant. If one of these variables is a constant, we can use the proportionality relation among the other two to solve the question orally.

Scenarios

When we say that in questions there are two scenarios, it requires some efforts on your part to identify the two scenarios. The following are some common instances of how to tune your thought process towards identifying the two scenarios ......

(Complete questions are not given, just the crucial info is given, so don't be too argumentative about the conclusions)

1.Today I travelled 20 % faster from home to office ...

The two scenarios are 'everyday' and 'today'. And while it is directly given that speeds are not constant, you need to realize the stated assumption that distance covered, home to office, will not change between 'everyday' and 'today' and thus, distance is constant

2. Two friends started simultaneously from their homes towards each other to meet ......

The two scenarios are that of the two individuals. Since they start simultaneously and they meet, both of them are travelling for the same time. Thus, the time is constant when we consider the case of the two individuals separately.

3. A train takes 10 secs to cross a pole and 15 seconds to cross a platform ......

While we will see this scenario in details later on, the two scenarios are obvious – one is 'train crossing pole' and other is 'train crossing platform'. And in the two scenarios, the speed of the train is going to be constant.

It will be on immense help to think on the lines of two scenarios. And then using proportionality. Most questions in the following text will be done by the traditional approach and then solved by the approach of using proportionality.

Distance being constant

Time is inversely proportional to speed, when distance is constant.

This is to say that, over a same distance, if the ratio of speeds is $a : b$, the ratio of the time taken will be $b : a$.

And this should be obvious, because over a same distance, if I double my speed (ratio of speeds 1 : 2), the time taken will be half (ratio of time 2 : 1).

If I travel at $\frac{1}{3}^{rd}$ the usual speed (ratio of speed 3 : 1), I would take thrice the time taken earlier (ratio of time 1 : 3)

If I reduce my speed to $\frac{3}{5}^{th}$ of the usual speed (ratio of speed 5 : 3), the time taken will be $\frac{5}{3}$ times the usual time (ratio of time 3 : 5).

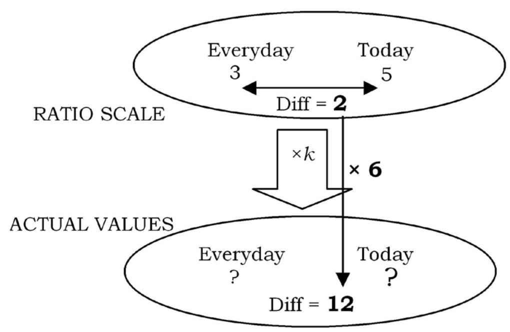
E.g. 4: From home to office, if I travel at $\frac{3}{5}^{th}$ of my usual speed, I am late by 12 minutes. Find the time that I take usually and the time taken at the reduced speed.

In such problems, 'late by 12 minutes' implies that 12 more minutes will be taken to travel the same distance, or in other words, the difference in the time taken at the usual speed and the reduced speed will be 12 minutes.

Had my usual speed been s, the reduced speed would be $\frac{3}{5}s$. Thus the ratio of usual speed to reduced speed would be 5 : 3. (From next problem onwards, this step will be done directly).

Since time is inversely proportional to speed, the ratio of the time is 3 : 5. We also know that the difference in the time taken will be 12 minutes.

Thus we are looking for two numbers that are in the ratio 3 : 5 and the difference between them is 12 minutes. In the chapter on ratios, we learnt an oral way to do these problems. The situation should be captured in your minds as follows:



Thus the usual time taken is 3 × 6 = 18 minutes and the time taken at reduced speed is 5 × 6 = 30 minutes.

E.g. 5: If a man walks at the rate of 5 kmph, he misses a train by 7 minutes. However, if he walks at the rate of 6 kmph, he reaches the station 5 minutes before the departure of the train. Find the distance to the station.

The ratio of speeds is 5 : 6 and since distance is constant, the ratio of the time taken will be in the ratio 6 : 5.

Missing the train by 7 minutes and reaching early by 5 minutes implies that the time taken at speed of 6 kmph is 12 minutes less than the time taken at speed of 5 kmph.

Thus, we need to find two numbers in ratio 6 : 5 with a difference of 12. Thus, a difference of 1 on the ratio scale is a difference of 12 in actual values. Hence the multiplying factor is 12. Thus time taken at 5 kmph is 6 × 12 = 72 minutes and at 6 kmph is 5 × 12 = 60 minutes.

Either of these speed and time combination can be used to find the distance. Since 60 minutes is 1 hour, it is easier to find the distance as 6 kmph × 1 hr = 6 km. Check that the same distance is found using the other combination of 5 kmph and 72 minutes.

Thus, the required distance is 6 km.

E.g. 6: A train meets with an accident and travels at $\frac{4}{7}^{th}$ of its regular speed hereafter and hence it reaches its destination 36 minutes late. How much time does the train take to reach its destination from the site of the accident had it traveled at its regular speed?

The two scenarios are that at 'regular speed' and that at 'decreased speed' between the distance from site of accident to destination.

Ratio of speeds in the two scenarios is 7 : 4. Ratio of time taken from site of accident to destination = 4 : 7. And the difference is given to be 36. Thus, a difference of 3 on the ratio scale is a difference of 36 in actual values. Hence the multiplying factor is 12. Thus, time taken at regular speed = 4 ×12 = 48 minutes.

E.g. 7: A train meets with an accident and travels at $\frac{4}{7}^{th}$ of its regular speed hereafter and hence it reaches its destination 36 minutes late. Had the accident occurred 30 kms further, the train would have been late by only 21 minutes. Find the regular speed of the train.

Let's say the point where the accident occurred was A when the train was late by 36 minutes and was point B when the train was late by 21 minutes. Let the destination be D.

Comparison over distance AD: The two scenarios are 'at regular speed' and 'at reduced speed'. Ratio of speeds 7 : 4. Ratio of time 4 : 7. Difference in time is 36 minutes. Thus, time taken at regular speed for the distance AD is 4 × 12 = 48 minutes.

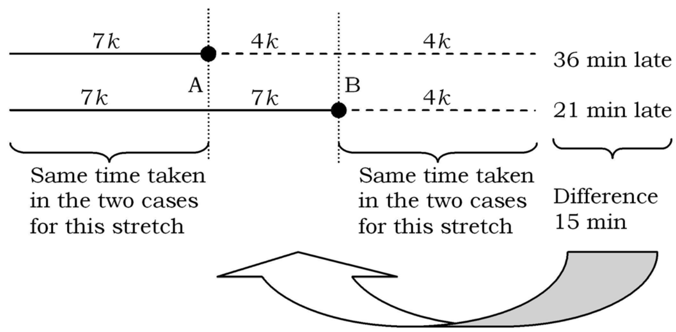
Comparison over distance BD: The two scenarios are 'at regular speed' and 'at reduced speed'. Ratio of speeds 7 : 4. Ratio of time 4 : 7. Difference in time, over this distance is 21 minutes. Thus, time taken at regular speed for the distance BD is 4 × 7 = 28 minutes.

At it's regular speed, the train takes 48 minutes to travel A to D and takes 28 minutes to travel B to D. Hence it must be taking 20 minutes to travel from A to B, a distance of 30 kms. Thus, it's regular speed = $\frac{30 \text{ km}}{20/60 \text{ hr}}$ = 90 kmph.

Shortcut:

While in the above solution we had the comparisons (at regular speed & at reduced speed) made twice, once over distance AD and once over distance BD. We could make do with just onne comparison as well.

Consider the stretch AB. When accident occurs at A, this stretch is travelled at reduced speed. And when accident occurs at B, this stretch is travelled at regular speed. And the difference in the time taken **just over this stretch** is 36 – 21 = 15 minutes. (At destination, why is the train less late, 21 mins, when accident occurs at B as compared to more late, 36 mins, when accident occurs at A? The train would have been lesser late because, over AB, it would have not have got late) See figure if this is not yet clear.



Comparing the two scenarios, ratio of speeds over AB is 4 : 7. Hence ratio of time will be 7 : 4 and difference in time taken over AB is 15 minutes. The multiplying factor to go from ratio scale to actual values will be 5. Thus, at regular speed it would take 4 × 5 = 20 minutes to cover AB, a distance of 30 km.

Time being constant

Distance is directly proportional to speed, when the time is same

This is to say, if time is constant and the ratio of speeds is $a : b$, the ratio of the distance covered will also be $a : b$.

The most common case of time remaining same would be when two persons, trains or objects start from two points simultaneously and meet each other. In this case the time that the two objects are traveling is the same for both the objects. So they will cover distances in proportion to their speeds.

E. g. 8: Two trains start simultaneously, one from Bombay to Kolkata and other from Kolkata to Bombay. They meet each other at Nagpur which is at a distance of 700 kms from Bombay. If the distance between Bombay and Kolkata is 1600 km, find the ratio of their speeds.

Since the trains started simultaneously, the time they have been traveling till they meet is equal. And hence the distance they cover will be in ratio of their speed. Since the train from Bombay has covered 700 km and the train from Kolkata has covered 1600 – 700 = 900 kms, the ratio of their speeds will be 700 : 900 i.e. 7 : 9.

E.g. 9: Two friends start walking towards each other with speeds in the ratio 3 : 4. When they meet it is found that the faster of them has covered 25 meters more than the slower. Find the distance that separated them initially if they are walking in opposite directions, but obviously towards each other.

Again in this case since they are walking for the same amount of time the ratio of the distance covered will also be 3 : 4. The distance given in the question, 25 meters, refers to the difference in the distances covered by them. Thus, a difference of 1 in the ratio scale corresponds to an actual distance of 25 i.e. multiplying factor will be 25.

We are required to find the distance separating them i.e. the sum of distances travelled by both. In the ratio scale this will be $3 + 4 = 7$ and the actual distance will be $7 \times 25 = 175$.

E.g. 10: A police-man starts chasing a thief. The ratio of the speeds of the thief and the policeman is 9 : 11 and when the policeman catches the thief it is found that the policeman has covered 60 meters more than the thief. How much distance did the police have to run to nab the thief?

Since the chase starts with both of them running simultaneously, from this point onwards to the time the police has caught the thief, they are running for same duration. Thus the distance covered will be proportional to their speeds. So we are searching for two distances in the ratio 9 : 11 and the difference being 60 meters i.e. 2 of the ratio scale corresponds to 60 mts, implying that the multiplying factor is 30. Hence distance run by police will be $11 \times 30 = 330$.

E.g. 11: In the movie Ghulam, Aamir is able to spot the approaching train when it is 2 km away. He has to run towards the train and reach the red kerchief hung on a pole 400 meters away from him before the train reaches the pole. How fast must Aamir run if the speed of the train is 36 kmph so that he just manages to reach the kerchief at the same time as the train reaches it?

From the point when the distance between them is 2000 m, Aamir has to cover a distance of 400 m and the train will cover the rest of the 1600 m i.e. the distance will be in the ratio 1 : 4. Since they are running for the same duration, the speed will be proportional to the distance covered. Since the speed of the train is 36 kmph, the speed of Aamir should be 9 kmph.

E.g. 12: In a race of 100 meters, A beats B by 10 meters and C by 20 meters. By how many meters does B beat C in the same race?

Most of the problem on race are based on the proportionality of speed and distance because in the case of race, the runners start running simultaneously and thus at any point of time (when the runners are yet running), they have been running for the same time.

When A has run 100 meters, B would have run 90 meters and C would have run 80 meters. Since this has happened in the same time, the ratio of the speeds of B and C is 9 : 8. Now we have to find the difference between B and C when B has run 100 meters. So again we are searching for difference between two numbers in the ratio 9 : 8 when the first of them is 100 meters. Since 9 of ratio scale corresponds to 100 m, 8 of the ratio scale will be equal to $\frac{8}{9} \times 100$ = 88.88 m. Thus, B beats C by i.e. 11.111... mts.

Speed being constant

Distance is directly proportional to time when speed is constant

At same speed, if the ratio of speed is $a : b$, the ratio of time will also be $a : b$.

By now, you would have got the hang of solving questions based on proportionality and thus we will solve just one example for this proportionality.

E.g. 13: A car overtakes an auto at point A at 9 am. The car reaches point B at 11 am and immediately turns back. It again meets the auto at point C at 11:30 am. At what time will the auto reach B?

Since the car takes 2 hours to travel AB and 0.5 hours to travel BC,
The ratio of the distances AB : BC will be 2 : 0.5 i.e. 4 : 1. (Distance is proportional to time)

Since C lies in between A and B, the ratio of the distance AC to CB will be 3 : 1.

The auto has traveled AC in 2.5 hours (from 9 am to 11:30 am). To travel CB (one-third distance of AC) he will take $\frac{2.5 \text{ hrs}}{3} = \frac{150 \text{ min}}{3}$ = 50 more minutes. Thus the auto will reach B at 12:40 pm.

Interesting Insight – Using AM & HM in proportionality relations

Quite often in questions we find that the given speeds (or time taken) are in an Arithmetic Progression. And if distance covered at the speeds is constant, then time taken (or speeds) will be inversely proportional i.e. they will be in Harmonic Progression.

For those who have forgotten, the Arithmetic Mean of a and b is $\frac{a+b}{2}$ and the Harmonic Mean of a and b is $\frac{2ab}{a+b}$.

Though it requires a little trained eyes to identify the above, it will be useful if you keep a watch for it. See the following data to realise that either time taken or speeds are in an Arithmetic Progression.

E.g.: A, B, C leave point P, one after the other in the given order, with equal time intervals between their departure. If all three simultaneously meet at Q, given that speed of A and C is 40 kmph and 60 kmph, find speed of B.

...... cont'd

The time taken by A, B, C over constant distance PQ will be of the type t, $t - x$ and $t - 2x$ i.e. in an AP. Thus, their speeds will be in a Harmonic Progression. The required speed will be the HM of 30 and 60 i.e. $\frac{2 \times 30 \times 60}{90}$ = 40 kmph

E.g.: If I travel at 10 kmph, I reach office at 10 am, if I travel at 15 kmph, I reach office at 10:30 am. At what speed should I travel so that I reach office at 10:15. Assume I leave home at same time and take the same route.

Leaving at same time and reaching at 10 am, 10:15 am and 10:30 am suggests that the time travelled are in AP. Thus, speeds are in HP and required speed is the HM of 10 & 15 i.e. $\frac{2 \times 10 \times 15}{25}$ = 12 kmph.

E.g.: Everyday I cover the distance from my home and office at a usual speed and take a certain time at the usual speed. When I increase my usual speed by 5 kmph, I take 10 minutes less than usual. If I reduce my usual speed by 5 kmph, I take 15 minutes more than usual. Find the distance from home to office.

This is the last question of previous exercise and you must have realised how time consuming it is to solve by equations.

Can you realise that if usual speed is s, then the three speeds are (s – 5), s and (s + 5) i.e. in AP? Thus, the time taken are in HP. The time taken as per the data in question is (t + 15), t and (t – 10) and hence we have $t = \frac{2 \times (t+15) \times (t-10)}{2t+5} \Rightarrow 2t^2 + 5t = 2t^2 + 10t - 300$

i.e. $5t = 300$ i.e. $t = 60$ mins.

Thus time taken are 75 mins, 60 mins and 50 mins. Speeds will be in ratio of $\frac{1}{75} : \frac{1}{60} : \frac{1}{50}$ i.e. 4 : 5 : 6. And we know the difference in speeds are 5 kmph. Thus speeds are 20 kmph, 25 kmph and 30 kmph. Now distance can be found using any combination of speed and time.

E.g.: In boats and streams, speed of boat travelling upstream, of boat alone and speed while travelling downstream will be $B - S$, B and $B + S$, again in AP. Thus the above scenario can also be used in this situation.

Exercise

8. Over a certain distance, if the speed had been 10% more, by how much percent less would the time taken be?

   1. 11% 2. 10% 3. 9.09% 4. 8.88% 5. Depends on distance

9. At its usual speed of 40 kmph, the bus covers its journey on schedule. But when its speed reduces to 35 kmph, it takes 15 more minutes than scheduled time. Find the distance of the journey.

   1. 50 km 2. 60 km 3. 70 km 4. 75 km 5. 80 km

10. A car covers a distance in 8 hours. Had the speed been increased by 4 kmph, the time taken would have reduced by 30 minutes. Find the distance.

    1. 360 km 2. 320 km 3. 400 km 4. 500 km 5. 480 km

11. Two swimmers start from same end, A, of a 200 meter long swimming pool. The faster one reaches the other end, B, turns back and meets the slower one at a distance of 175 meters from the starting end. Where will the slower one be when faster one reaches starting point.

1. Mid - way	2. 111 mts from A	3. 89 mts from A
4. 87.5 mts from A	5. 112.5 mts from A	

12. A train met with an accident 150 km from its originating station. It completed the remaining journey at half of the usual speed and reached 1 hour late at the destination station. Had the accident taken place 30 km later, it would have been only half an hour late. Find the distance between the two stations.

1. 180 km 2. 200 km 3. 250 km 4. 400 km 5. None of these

13. A is twice as fast as B and B is thrice as fast as C. If B takes 18 minutes to cover a certain distance, find the difference in the time taken by A and C to cover the same distance.

1. 30 mins 2. 36 mins 3. 40 mins 4. 45 mins 5. 54 mins

14. Two cars separated by 600 meters start moving towards each other (in opposite direction) with speeds of 25 m/s and 35 m/s and meet at point A. If they moved in same direction with faster car chasing slower, they would meet at B. Find distance AB.

1. 1750 mts 2. 1800 mts 3. 2100 mts 4. 2400 mts 5. 2450 mts

15. What should the length of a race be so that two drivers with speeds of 22 m/s and 25 m/s reach the end point simultaneously even though the slower one had a head-start of 6 minutes (i.e. the slower one starts racing first and after 6 minutes the faster one starts)?

1. 44 km 2. 50 km 3. 55 km 4. 66 km 5. 75 km

16. What should the length of a race be so that two drivers with speeds of 22 m/s and 25 m/s reach the end point simultaneously even though the slower one had a head-start of 6 meters (i.e. the faster one starts from the starting point but the slower one starts from 6 meters ahead)?

1. 50 mts 2. 50 km 3. 44 mts 4. 44 km 5. 550 mts

Relative Speed

When two objects are moving simultaneously with speeds S_1 and S_2, the speed of any object when observed from the other object's perspective is called the relative speed. And it is distinct from S_1 or S_2.

E.g.: Consider yourself to be sitting in a moving train and another train passes your train. If the other train is in opposite direction to your train, the speed of the other train appears to be far far more than it actually is. And if the other train is in the same direction as yours, it just appears to be inching ahead of your train very very slowly. This observed/perceived speed is called as relative speed and is calculated as:

If two objects are moving with speeds S_1 and S_2, their relative speed is

$S_1 + S_2$, if they are moving in opposite direction and

$S_1 - S_2$, if they are moving in same direction.

Relative speed is usually considered when one has to find the time taken to meet or catch and it can be found as follows:

$$\text{Time taken to meet/catch} = \frac{\text{Initial distance separating them}}{\text{Relative Speed i.e. } S_1 \pm S_2}$$

E.g. 14: A thief escapes from a prison at 2 pm and travels away at a speed of 30 kmph. The police realize the escape at 3:30 pm and start the chase then at a speed of 40 kmph. At what time will the police catch the thief? At what distance from the prison is the thief caught?

Approach 1: Using Relative Speed

When the chase starts i.e. at 3:30 pm, the thief has already run 30 × 1.5 = 45 kms. Thus, this is the distance that separated the police and the thief and so the time taken to catch is $\frac{45}{40-30} = \frac{45}{10}$ = 4.5 hours. This 4.5 hours is measured from the time chase starts i.e. from 3:30 and thus the thief is caught at 8 pm.

The distance run by the police is 40 kmph × 4.5 hours = 180 kms and so the thief is caught at a distance of 180 kms from the prison.

Approach 2: Using Proportionality

Method 1: Considering distance constant

Thief escapes from police station and say is caught at point X. The police also run the same distance, from police station to X. Thus, the distance run by police and thief is the same. Hence the time for which they are running will be inversely proportional to their speeds. Since ratio of speeds of police and thief is 4 : 3, time for which they are running will be 3 : 4. Also the difference in the time for which they are running is 1½ hour (thief starts running 1½ hours earlier). Thus, police and thief have been running for 3 × 1.5 = 4.5 hrs and 4 × 1.5 = 6 hrs respectively. And so the thief is caught at 3:30 pm + 4.5 hrs i.e. 8 pm (or 2 pm + 6 hrs i.e. 8 pm)

Method 2: Considering time as constant

If we consider the time interval for which the chase is on i.e. from the time police start chasing to the time thief is caught, then the time run will be the same. However in this case the distance run is different (because we are considering the event from 3:30 pm when thief has a head-start of 30 × 1.5 = 45 km).

With time being constant, the distances run will be proportional to speed. Thus, ratio of speed and distance will be 4 : 3. And we know that since 3:30, the police have to run a distance of 45 km more than the thief. Thus, 1 of ratio scale corresponds to 45 km and police will cover a total of 4 × 45 = 180 km in catching the thief. Time taken for this will be $\frac{180}{40}$ = 4.5 hrs.

E.g. 15: Navjivan Express from Ahmedabad to Chennai leaves Ahmedabad at 6:30 am and travels at 50 kmph towards Baroda situated 100 kms away. At 7:00 am, Howrah-Ahmedabad Express leaves Baroda towards Ahmedabad and travels at 40 kmph. At 7:30 Mr. Shah, the traffic controller at Baroda realizes that both the trains are running on the same track. How much time does he have to avert a head-on collision between the two trains?

Considering the reference time to be 7:30 am, we would need to find the distance between the two trains at 7:30 am. From 6:30 to 7:30, the Navjivan Express would have traveled 50 kms and from 7 to 7:30, the Howrah-Ahmedabad would have traveled 20 kms. Thus the distance separating the two trains at 7:30 would be 100 – 50 – 20 = 30 kms.

Approach 1: Using Relative Speed

Time to meet (collide) from 7:30 am is $\frac{30}{50+40} = \frac{30}{90} = \frac{1}{3}$ hrs i.e. 20 minutes.

Approach 2: Using proportionality

Since 7:30 to the time they meet, the two trains are travelling for same time. Thus, the distances covered will be in the same ratio as their speeds i.e. 5 : 4. And we know that the total distance covered by the two trains in this time is 30 km. Thus the trains individually cover, $\frac{5}{9} \times 30 = \frac{50}{3}$ km and $\frac{4}{9} \times 30 = \frac{40}{3}$. The time taken for this is $\frac{50/3 \text{ km}}{50 \text{ kmph}} = \frac{1}{3}$ hr (or $\frac{40/3 \text{ km}}{40 \text{ kmph}} = \frac{1}{3}$ hr)

E.g. 16: A starts from X at 9:00 am and reaches Y at 1:00 pm. B starts from Y at 9:00 am and reaches X at 3 pm. At what time do the two meet?

Approach 1: Using Relative Speed

Assuming the distance between X and Y to be d, since A and B take 4 hours and 6 hours respectively to cover the distance, their speeds are $\frac{d}{4}$ and $\frac{d}{6}$ respectively. Thus the time taken to meet will be $\frac{d}{\frac{d}{4}+\frac{d}{6}} = \frac{1}{\frac{1}{4}+\frac{1}{6}} = \frac{4\times 6}{6+4} =$ 2.4 hrs, after 9 am i.e. at 11:24 am.

Short-cut:

Since in the above solution, we see that the answer is independent of the distance between X and Y, we can assume any value for the distance such that our calculations are easy. Since the time taken by A and B are 4 hours and 6 hours, we take the distance XY to be the LCM of 4 and 6 i.e. 12 km. So the speeds of A and B are 3 kmph and 2 kmph respectively.

Now time taken to meet = $\frac{12}{3+2} = \frac{12}{5}$ = 2.4 hrs, just as found above.

Approach II: Use of proportionality

Considering the case when individually both A and B cover the entire distance XY. A takes 4 hours to cover XY and B takes 6 hours to cover XY. Since the distance covered is same, the ratio of their speed will be in inverse proportion to the time taken by each. Thus, ratio of speed of A and B is 3 : 2.

Now considering that they start simultaneously from X and Y respectively and meet. In this case, the time for which they are travelling is the same. So the distance that they cover will be in ratio of their speeds. Thus, A will cover $\frac{3}{5}^{th}$ of XY (and B will cover $\frac{2}{5}^{th}$ of XY).

A takes 4 hours to cover complete XY and hence will take $\frac{3}{5}^{th} \times 4 \text{ hrs} = \frac{12}{5}$ = 2.4 hrs to cover $3/5^{th}$ of XY.

E.g. 17: A starts from X at 9:00 am and reaches Y at 1:00 pm. B starts from Y at 10:00 am and reaches X at 4 pm. At what time do the two meet?

This question is very similar to the above question except that both A and B are not starting simultaneously. A takes 4 hours to travel XY and B takes 6 hours to travel XY. Assuming the distance to be 12 km (for reasons given in the above solution), the speeds of A and B are found to be 3 kmph and 2 kmph.

Since A starts 1 hour earlier than B starts, by 10 am he would have covered 3 kmph. So when both of them are moving (i.e. when the relative speed is 3 + 2 = 5 kmph), the distance separating them is 12 – 3 = 9 km. Thus time taken to meet = $\frac{9}{5}$ = 1.8 hours, after 10 am i.e. at 11:48 am.

If one is wondering how can we assume any distance, just see the following expression and realize that the answer is independent of the distance. Assuming the distance XY as d, speeds of A and B are $\frac{d}{4}$ & $\frac{d}{6}$

$$\text{Time taken to meet} = \frac{\text{distance separating them}}{\text{relative speed}} = \frac{d - \frac{d}{4} \times 1}{\frac{d}{4} + \frac{d}{6}}$$

Exercise

17. The distance between two cities A and B is 330 kms. A train starts from A at 8 am and travels towards B at 60 km/hr. Another train starts from B at 9 am and travels towards A at 75 kmph. At what time do the two trains meet?

 1. 10 a.m. 2. 11 a.m. 3. 12 noon 4. 1 p.m. 5. 2 p.m.

18. A and B can cover the distance XY in 12 and 8 hours respectively. If both of them start from opposite ends of XY towards each other, after how much time will they meet?

 1. 4.8 hrs 2. 5 hrs 3. 4 hrs 4. 10 hrs 5. Cannot be determined

19. A thief escapes from city A at 2 pm and flees towards city B at 40 kmph. At 3 pm, the police realize the escape and start chasing the thief at 50 kmph. Simultaneously, a police team from station B also starts towards city A to apprehend the thief at a speed of 60 kmph. What should be the distance between A and B such that both the police team nab the thief at the same time?

 1. 200 km 2. 240 km 3. 400 km 4. 440 km 5. 480 km

20. Two persons A and B are separated by a certain distance. If they move towards each other they meet after 20 minutes but if A chases B, they meet after 100 minutes. Find the ratio of the speeds of A and B.

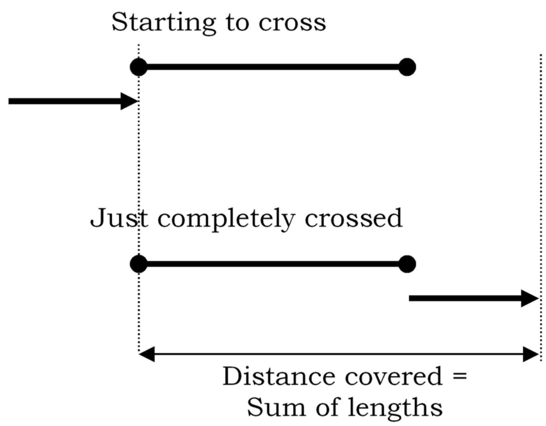
 1. 5 : 1 2. 1 : 5 3. 4 : 3 4. 2 : 3 5. 3 : 2

21. In a foggy night, the visibility is limited to just 300 mts. Exactly 2 minutes after the driver of a speeding car could see a bus ahead, the driver of the bus sees the car vanishing from his sight, ahead. If the speed of the bus is 36 kmph, what is the speed of the car?

 1. 54 kmph 2. 45 kmph 3. 42 kmph 4. 40 kmph 5. 38 kmph

Train crossing a pole/man/platform/bridge/train

Whenever one object of finite length crosses another object of finite length, in crossing the object completely it covers a distance equal to the sum of the lengths. The distance to be covered is independent of the direction in which it is crossing i.e. whether the two objects are crossing each other in opposite directions or if one is overtaking the other i.e. in same direction, the distance to be covered for one to completely cross the other is the sum of the lengths.



In the problems asked, usually a train is crossing either a pole or a man (stationary or walking) or a platform or bridge or another train (obviously moving). For all such cases you could use the following with the modifications as noted below:

Time taken to cross = $\dfrac{L_1 + L_2}{S_1 \pm S_2}$

If one of the object is a pole or a man, its length will be 0 (zero)

If one of the object is stationary (e.g. pole, platform, bridge), its speed will be 0 (zero) and the relative speed will just be the speed of the moving object.

E.g. 18: A train running at 72 kmph crosses a telephone pole in 7 sec. What is the length of the train?

Since a telephone pole is a stationary object of negligible length, the distance the train covers is just its own length.

Converting 72 kmph into m/s, since 72 is 4^{th} multiple of 18, speed in m/s is 4^{th} multiple of 5 i.e. 20 m/s

Distance = 20 × 7 = 140 m

So length of the train = 140 m

E.g. 19: A train crosses 2 platforms of length 400 m and 600 m in 6 seconds and 8 seconds respectively. What is the length of the train?

Approach 1: Using formula

Let the length of the train be x.

Then, $\frac{400+x}{S_t} = 6$ and $\frac{600+x}{S_t} = 8$

Since we want to find x, eliminating S_t, we have $\frac{400+x}{6} = \frac{600+x}{8} \Rightarrow$

$2x = 400$ i.e. $x = 200$.
Hence the length of the train is 200 m.

Approach 2: Using proportionality

In the two scenarios 'crossing platform of 400 m in 6 sec' and 'crossing platform of 600 m in 8 sec', the speed is the same. Hence distance will be proportional to time taken. The distances covered are 400 + x and 600 + x i.e. a difference of 200. And the time taken are 6 sec & 8 sec. Thus ratio of time is 3 : 4. This will also be the ratio of distances and we know that the difference in distances is 200. Thus, the distances covered in the two scenarios is 600 m and 800 m. Thus, length of train is 200 m.

E.g. 20: A train crosses two persons who are walking at 2 kmph and 4 kmph, in the same direction in which the train is going, in 9 and 10 seconds respectively. Find the length of the train.

Approach 1: Using formula

If the length (in kms) and speed (in kmph) of the train is L and S_t respectively, we have

$\frac{L}{S_t - 2} = \frac{9}{3600}$ and $\frac{L}{S_t - 4} = \frac{10}{3600}$

i.e. $400L = S_t - 2$ and $360L = S_t - 4$

Subtracting $40L = 2$ i.e. $L = \frac{1}{20}$ km $= \frac{1000}{20}$ m = 50 m.

Approach 2: using Proportionality

In the two scenarios, 'train crossing man walking at 2 kmph' and 'train crossing man walking at 4 kmph', the distance covered will be the length of the train itself i.e. will be same in both the cases. Thus, ratio of speed will be inverse of ratio of time taken. Since ratio of time taken is 9 : 10, ratio of speed will be 10 : 9. Further we also know that the difference in the speeds in the two scenario will be 2 kmph (speeds will be S – 2 and S – 4). Thus, the speeds in the two scenario are 20 kmph and 18 kmph.

At 18 kmph i.e. 5 m/s, distance covered in 10 sec will be 50 m. This has to be the length of the train.

Exercise

22. Two trains, each 100 meters long take 60 seconds to cross each other if traveling in the same direction and 10 seconds to cross each other when traveling in opposite direction. Find the speed of the faster train.

    1. 30 kmph 2. 35 kmph 3. 42 kmph 4. 50 kmph 5. 84 kmph

23. Two trains traveling at same speed take 10 seconds and 15 seconds respectively to cross a telegraph post. Find the time taken by the trains to cross each other if traveling in opposite directions.

    1. 6 sec 2. 12 sec 3. 12.5 sec 4. 25 sec 5. Cannot be determined

24. Two trains running in opposite directions cross a man standing on a platform in 27 and 17 seconds respectively. They cross each other in 23 seconds. Find the ratio of the speeds of the trains.

    1. 2 : 3 2. 1 : 3 3. 1 : 2 4. 3 : 4 5. Cannot be determined

25. A train traveling at 72 kmph crosses a platform in 30 seconds and a man standing on the platform in 18 seconds. What is the length of the platform in meters?

    1. 100 2. 125 3. 180 4. 200 5. 240

26. How much time will an express train of length 150 meters and running at a speed of 75 kmph take to cross a man walking at 5 m/s inside a passenger train of length 200 meters and running at 15 kmph in a direction opposite to that of the express train. The man is walking inside the train and in opposite direction to that of the train. (Assume the man does not reach the end of his train meanwhile.)

    1. 5 sec 2. 6 sec 3. 7.5 sec 4. 8 sec 5. 10 sec

27. Two trains running at 20 m/s and 35 m/s cross the same tunnel in 10s and 12s respectively. What is the difference in the length of the two trains?

    1. 84 mts 2. 110 mts 3. 60 mts 4. 220 mts 5. 165 mts

28. Two trains are traveling in the same direction at 50 kmph and 30 kmph. The faster train crosses a man sitting in the slower train in 18 seconds. Find the length of the faster train.

    1. 80 mts 2. 100 mts 3. 120 mts 4. 125 mts 5. 150 mts

Boats and Streams

When a boatman is rowing in still water, say a lake, he would be moving at a speed at which he can row. This speed is called the speed of boat in still water or simply speed of boat. But consider the same boatman in a stream. Because of the current he is either aided (if rowing in the direction of the stream, this is called Downstream) or will be opposed (if rowing against the stream, called Upstream).

If the speed of boatman in still water is *B* and the speed of the stream is *S*, we have

Downstream Speed (D) = B + S

Upstream Speed (U) = B – S

Do not confuse with relative speed

In relative speed we saw that when two objects travel in same direction with speeds S_1 and S_2, the relative speed is $S_1 - S_2$. In the case of boats and streams, when boat is travelling in the direction of stream, the downstream speed of the boat is taken as B + S. Since they are in same direction, should we not consider relative speed to be B – S? No.

In case of relative speeds, the two objects are moving independently i.e. say police running to catch thief, the speed of police does not get 'transferred' or affects the speed of the thief. This is a case of relative speeds.

Whereas in the case of Boats and Streams, the speed of the Stream gets 'transferred' to the speed of the boat, the boat is travelling 'on' the stream. This is not a case of relative speed. Here the stream is 'aiding' the boat, unlike the case of police and thief where neither 'aids/ hinders' the other.

If the speed of boatman is lesser than the speed of the current or stream, the upstream speed will be negative i.e. he is trying to row upstream, but rather than move in that direction, he is taken in opposite direction by the stream. But such situations do not occur in math problems on this topic.

The above relations can be modified for cases when the effective downstream speed, D and upstream speed, U is given and we are supposed to find B and S.

$$B = \frac{D+U}{2};\ S = \frac{D-U}{2}$$

E.g. 21: A boat covers a distance of 16 km in 2 hours when rowing downstream and in 4 hours if rowing upstream. What is the speed of the boat in still water?

Speed Upstream = $\frac{16}{2}$ = 8 km/hr and Speed Downstream = $\frac{16}{4}$ = 4 km/hr

From the above formula, speed of boat in still water = $\frac{D+U}{2} = \frac{8+4}{2}$ = 6 km/hr

E.g. 22: The rowing speed of a man in still water is 7.5 kmph. In a river flowing at 1.5 kmph, it takes the same boatman 50 minutes to row a certain distance and come back. Find the distance.

Here Speed Downstream = 7.5 + 1.5 = 9 km/hr and Speed Upstream = 7.5 – 1.5 = 6 km/hr

Approach 1: Formulaic approach

Let the required distance be x. Equating the time taken, $\frac{x}{9} + \frac{x}{6} = \frac{50}{60}$ hrs

Solving we get, $x = 3$. Hence the place is 3 km away.

Approach 2: Use of proportionality.

In going upstream and downstream, the distance covered is same. Hence time taken are in inverse proportion to their speeds. Ratio of upstream and downstream speeds is 6 : 9 i.e. 2 : 3. Thus, ratio of time travelling upstream and downstream is 3 : 2. And we know the total time is 50 minutes. Thus time travelled upstream is 30 minutes and time travelled downstream is 20 minutes. Now the distance can be found using speed × time i.e. 6 kmph × 30 min = 6 kmph × ½ hr = 3 km (it could also be found using downstream speed and time i.e. 9 kmph × 20 min = 9 kmph × 1/3 hr = 3 km)

Anatomy of standard questions

Usually in the case of questions on boats and streams, the data involves a boat travelling some distance downstream and some distance upstream. Either the total time is given or the total distance is given. Thus, we should have the flexibility to form equations regarding the total distance or the total time.

Equation for total time: $T_{down} + T_{up}$ = Total Time

i.e. $\frac{D_{down}}{B + S} + \frac{D_{up}}{B - S}$ = Total Time

Equation for total distance: $D_{down} + D_{up}$ = Total Distance

i.e. $(B + S) \times T_{down} + (B - S) \times T_{up}$ = Total Distance

Familiarity with the above and ability to quickly realise which of the two equations can be formed from the given data will go a long way in building your speed of solving.

While trying the following questions of the exercise, see if you can identify that invariably most questions are of the above form.

E.g. 23: A boatman rows to a place 48 km distant and back in 14 hours. He finds that he can row 4 km with the stream in the same time as 3 km against the stream. Find the rate of the stream.

Since the time taken to row 4 km downstream is same as the time taken to row 3 km upstream, $\frac{4}{B+S} = \frac{3}{B-S} \Rightarrow$ B = 7S

Since the boatman takes 14 hours to row to a place 48 km distant and come back, $\frac{48}{B+S} + \frac{48}{B-S} = 14$

Substituting B = 7S, we have $\frac{48}{8S} + \frac{48}{6S} = 14 \Rightarrow \frac{6}{S} + \frac{8}{S} = 14 \Rightarrow \frac{14}{S} = 14 \Rightarrow$

S = 1.

Approach 2: Using Proportionality

Consider the statement, 'he can row 4 km with the stream in the same time as 3 km against the stream'. Obviously the time is constant and thus, downstream speed and upstream speed are in the ratio of distances covered i.e. 4 : 3.

Next, consider travelling 48 km downstream and 48 km upstream. Again since distance is constant, time taken will be inversely proportional to speed i.e. time taken to travel downstream and that taken to travel upstream are in ratio 3 : 4. And we know that the total time taken is 14 hours. Thus, 6 hours is taken to travel 48 km downstream (speed = 8 kmph) and 8 hours is taken to travel 48 km upstream (speed = 6 kmph). Thus, speed of stream $= \frac{D-U}{2} = \frac{8-6}{2} = 1$

Exercise

29. The speed of a motor boat itself is 20 km/h and the rate of flow of the river is 4 km/h. Going downstream, the boat went 120 km. What distance will the boat cover during the same time going upstream?

    1. 40 km 2. 50 km 3. 60 km 4. 80 km 5. 90 km

30. A motorboat went down the river for 14 km and then up the river for 9 km. It took a total of 5 hours for the entire journey. Find the speed of the river flow if the speed of the boat in still water is 5 kmph.

    1. 1 kmph 2. 1.5 kmph 3. 2 kmph 4. 2.5 kmph 5. 3 kmph

31. In a stream that is running at 2 kmph, a man goes 10 km upstream and comes back to the starting point in 55 minutes. Find the speed of the boat in still water.

    1. 16 kmph 2. 18 kmph 3. 20 kmph 4. 22 kmph 5. 24 kmph

32. A boat takes a total of 6 hours to row 8 kms downstream and to return back to the starting point. If speed of the boat is 3 kmph, for how much time was the boat moving downstream?

    1. 1 hr 2. 2 hr 3. 3 hr 4. 4 hr 5. 5 hr

33. A man rows for 3 hrs downstream and then for 3 hrs upstream. In this whole process he covers a distance of 12 kms. If the speed of the stream is 1 kmph, for how much more time will he have to row upstream to reach the starting point?

    1. 6 hr 2. 5 hr 3. 4 hr 4. 3 hr 5. 2 hr

34. A boat travels from point A to point B upstream and returns from point B to point A downstream. If the round trip takes the boat 5 hours and the distance between point A and point B is 120 kms and the speed of the stream is 10 km/hr, how long did the upstream journey take?

    1. 2.5 hrs 2. 3 hrs 3. 3.5 hrs 4. 4 hrs 5. 4.5 hrs

35. A man can row 50 km upstream and 72 km downstream in 9 hours. He can also row 70 km upstream and 90 km downstream in 12 hours. Find the rate of current.

    1. 2 kmph 2. 3 kmph 3. 4 kmph 4. 6 kmph 5. 8 kmph

36. I travel for 2 hours downstream and then for 2 hours upstream. I am yet 16 kms away from the starting point. Find the speed of the flow of the river.

    1. 8 kmph 2. 6 kmph 3. 5 kmph 4. 4 kmph 5. Cannot be determined

Average Speed

During a journey, different parts of the journey could be covered at different speeds. Average speed is that uniform speed at which the total distance could be covered in the same time as it took in the actual case.

Thus, $\text{Average Speed} = \dfrac{\text{Total Distance}}{\text{Total Time}}$

It is not necessarily the arithmetic mean of different speeds, as seen in the following example:

E.g. 24: Atul covered 20 kms at a speed of 30 kmph and next 30 kms at a speed of 90 kmph. Find his average speed.

The total distance that Atul covered was 20 + 30 = 50 kms. To find the average speed, we would also need to find the total time take. This can be found by adding the individual time taken over the two stretches.

Total time taken = $\dfrac{20}{30}+\dfrac{30}{90}=\dfrac{2}{3}+\dfrac{1}{3}$ = 1 hr.

Thus average speed = $\dfrac{50\text{ km}}{1\text{ hr}}$ = 50 kmph

Please note that 50 kmph is not the arithmetic mean of 30 kmph and 90 kmph. Whereas in the following example the average speed turns out to be the arithmetic mean...

E.g. 25: Atul covered one fourth of a distance at a speed of 30 kmph and rest of the distance at a speed of 90 kmph. Find his average speed.

Since the distance is broken into one-fourth and three-fourth, let's assume the total distance to be $4d$

To find the average speed, we would also need to find the total time take. This can be found by adding the individual time taken over the two stretches.

Total time taken = $\dfrac{d}{30}+\dfrac{3d}{90}=\dfrac{6d}{90}=\dfrac{d}{15}$ hrs

Thus average speed = $\dfrac{4d\text{ km}}{d/15\text{ hr}}$ = 60 kmph

In this case the average speed 60 kmph is the arithmetic mean of 30 kmph and 90 kmph.

Thus the average speed need not necessarily be equal to the arithmetic mean of the different speeds. But it can also be equal to the arithmetic mean. Later we will decipher when is it going to be the arithmetic mean.

E.g. 26: Atul travelled for one fourth of the total time of a journey at a speed of 30 kmph and rest of the time at a speed of 90 kmph. Find his average speed.

In this case the journey is broken into stretches of time, one-fourth of time and three-fourth of time. So let us assume the total time as $4t$.

To find the average speed, we would also need to find the total distance covered. This can be found by adding the individual distances covered over the two stretches.

Total distance covered = $30 \times t + 90 \times 3t = 300t$

Thus, average speed $\frac{300t}{4t} = 75$

Please compare e.g. 25 and 26, while the numbers used were exactly similar, the average speed are different. Also in the earlier example distance was assumed as $4d$ and in the latter one time was assumed as $4t$. These two examples basically cover most of the types of the questions in this topic.

Stretches in terms of distance or time

As seen in the above solved examples, when "stretches" of the journey are traveled at different speeds, the "stretches" could be expressed in terms of distance or in terms of time:

"Atul traveled $\frac{1}{3}^{rd}$ of a distance at a speed of 30 kmph and the rest of the distance at a speed of 60 kmph" is different from the data "Atul traveled for $\frac{1}{3}^{rd}$ of the time at a speed of 30 kmph and the rest of the time at a speed of 60 kmph"

And the expression to find the average speed visually appears different but is essentially $\frac{\text{Total Distance}}{\text{Total Time}}$

Stretches in terms of distance:

If d_1, d_2, d_3, ... are the distances run at speeds s_1, s_2, s_3, ... respectively,

Average speed = $\dfrac{d_1 + d_2 + d_3 + ...}{\frac{d_1}{s_1} + \frac{d_2}{s_2} + \frac{d_3}{s_3} + ...}$

The numerator is the total distance covered and denominator is the total time taken and is found by adding the time taken in each stretch.

Stretches in terms of time:

If one travels for t_1, t_2, t_3, ... hours at speeds s_1, s_2, s_3, ... respectively,

Average speed = $\dfrac{(s_1 \times t_1) + (s_2 \times t_2) + (s_3 \times t_3) + ...}{t_1 + t_2 + t_3 + ...}$

The numerator is the total distance traveled and is found by adding the distances traveled in each time interval. The denominator is the total time taken.

In certain cases, you can have the liberty to assume variable for distances or for times (see next solved example). Looking at the two expressions, assuming variables for time is always going to be lesser on calculations.

E.g. 27: I travel two different stretches at speeds of 25 kmph and 40 kmph such that my average speed for the entire journey is 30 kmph. What fraction of the entire distance was travelled at 25kmph?

Approach 1: Assuming distances

Since the question requires us to find a fraction of distance, a logical way would be to assume the distance travelled at 25kmph and 40 kmph as d_1 and d_2.

Writing the term for average speed, $\dfrac{d_1 + d_2}{\dfrac{d_1}{25} + \dfrac{d_2}{40}} = 30$. This expression would involve more calculation than necessary. Cross-multiplying, $d_1 + d_2 = \dfrac{6}{5}d_1 + \dfrac{3}{4}d_2$. Multiplying both sides by 20,

$20d_1 + 20d_2 = 24d_1 + 15d_2$ i.e. $4d_1 = 5d_2$ i.e. $\dfrac{d_1}{d_2} = \dfrac{5}{4}$.

Thus, the fraction of distance covered at speed of 25 kmph is $\dfrac{5}{9}$.

Approach 2: Assuming time

Even though the question requires us to find a fraction of total distance, a lesser-calculation intensive approach would be to assume time travelled at 25 kmph and 40 kmph as t_1 and t_2.

Writing expression for average speed, $\dfrac{25t_1 + 40t_2}{t_1 + t_2} = 30 \Rightarrow \dfrac{t_1}{t_2} = \dfrac{10}{5} = \dfrac{2}{1}$

Now the ratio of distances can be found using D = S × T as $\dfrac{d_1}{d_2} = \dfrac{25 \times 2}{40 \times 1} = \dfrac{5}{4}$

Thus, the fraction of distance covered at speed of 25 kmph is $\dfrac{5}{9}$.

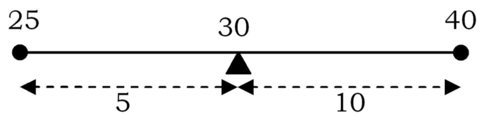
Average Speed as Weighted Average of speed.

Average speed as the term suggests has to involve an average. And indeed average speed is the average of the individual speeds, just that it is a case of weighted average and <u>the weight for each speed is the **time** traveled at that speed</u>.

Looking back at the expression, Average speed = $\dfrac{(s_1 \times t_1) + (s_2 \times t_2) + (s_3 \times t_3) + \ldots\ldots}{t_1 + t_2 + t_3 + \ldots\ldots}$,

one can relate the expression to that of finding the weighted average.

Thus, looking back at the just solved example, we are given that the average of 25 kmph and 40 kmph is 30 kmph. Thus, we could have just alligation/scale approach and found the weights visually as:



Thus, the weights would be 2 : 1. However what is important is to understand that this is the ratio of the time travelled at the individual speeds and not the distances covered.

Average speed is always weighted average of speeds with time being the weights

Even when the journey is given in stretches of distances covered, while the expression may appear different, the behind-the-scene-picture is that we are finding the weighted average of the speeds with time being the weights.

Consider solved e.g. 24 reproduced here again: Atul covered 20 kms at a speed of 30 kmph and next 30 kms at a speed of 90 kmph. Find his average speed.

We found average speed as $\frac{\text{Total Distance}}{\text{Total Time}} = \frac{20+30}{\frac{20}{30}+\frac{30}{90}} = \frac{50}{\frac{2}{3}+\frac{1}{3}}$ = 50 kmph.

This expression may visually not be similar to the expression of finding the weighted average. One could yet have used the funda of weighted averages by first finding the ratio of time taken in the two stretches. The time taken in the two stretches would be $\frac{20}{30}$ & $\frac{30}{90}$ i.e. $\frac{2}{3}$ & $\frac{1}{3}$ i.e. a ratio of 2 : 1. Now the average speed is $\frac{30\times2+90\times1}{2+1} = \frac{150}{3}$ = 50 kmph, same as obtained earlier.

Why are the weights not the distances?

As discussed in the chapter of weighted average the terms in brackets while finding $\frac{(C_1\times w_1)+(C_2\times w_2)}{(w_1+w_2)}$ have to have a meaning. S × D has no meaning whatsoever. Further when we consider S × T, the numerator becomes the total distance travelled and the denominator the total time and thus the expression is that of speed.

Now that we know that average speed is the weighted average of the individual speed and the weights are the time, we can answer the question – 'When would average speed be equal to the arithmetic mean of the speeds?'. Obviously, when the weights are all equal. Thus, if equal stretches of time are travelled at different speed, the average speed will be equal to the arithmetic mean of the speed. Recall e.g. 25. one fourth of a distance at a speed of 30 kmph and rest of the distance at a speed of 90 kmph implies that the ratio of time will be $\frac{d}{30}:\frac{3d}{90}$ i.e. 1 : 1. And hence the average speed was found to be the arithmetic mean of 30 and 90 i.e. 60 kmph.

This interpretation of the average speed being equal to the weighted average of the speeds with the weights being time, can be used to quickly solve few questions of average speed as follows:

E.g. 28: A man travels $\frac{1}{3}^{rd}$ of a distance at a speed of 45 kmph, $\frac{1}{2}$ the distance at 60 kmph and rest of the distance as speed of 36 kmph. Find his average speed for the entire journey.

If total distance is taken to be 6, rest of the distance will be 6 – 2 – 3 = 1.

Rather than calculate, $\dfrac{6}{\frac{2}{45}+\frac{3}{60}+\frac{1}{36}}$, which is going to be tedious (try it),

a better approach would be to find the ratio of time taken and then use weighted averages …

Ratio of distance is 2 : 3 : 1 and ratio of speeds is 15 : 20 : 12. Thus, ratio of time taken will be $\frac{2}{15}:\frac{3}{20}:\frac{1}{12}$ i.e. 8 : 9 : 5

Average speed = $\dfrac{45\times 8+60\times 9+12\times 5}{8+9+5}=\dfrac{360+540+180}{22}=\dfrac{1080}{22}$ = 49.09

E.g. 29: A man covers equal distances at a speed of 30 kmph and 60 kmph. Find his average speed.

Since the distances are equal, assuming them to be d, the average speed = $\dfrac{2d}{\frac{d}{30}+\frac{d}{60}}=\dfrac{2d\times 60}{3d}$ = 40.

Special Case: Two stretches of equal distances, Harmonic Mean

A special case is when <u>two equal distances</u> are traveled at two different speeds, say u and v.

In this case, the average speed is $\dfrac{d+d}{\frac{d}{u}+\frac{d}{v}}=\dfrac{2}{\frac{1}{u}+\frac{1}{v}}=\dfrac{2uv}{u+v}$

Remember to use the formula only when two <u>equal</u> distances are run at speeds of u and v. Also remember how the formula is arrived at to understand why you might get weird results using the formula in some cases (next solved e.g.)

Harmonic Mean

The expression $\dfrac{2uv}{u+v}$ is the Harmonic mean of u and v. Thus, when equal distances are run at two speeds, the average speed is the Harmonic Mean (HM) of the two speeds. And when equal time durations are run at two speeds, the average speed is the Arithmetic Mean (AM) of the two speeds. And for any two positive numbers, $\mathrm{HM}\le\mathrm{AM}$. Thus, when equal distances are run at two speeds, the average speed is always less than the Arithmetic Mean of the two speeds.

E.g. 30: A man covers equal distances at a speed of 30 kmph and x kmph such that his average speed for the entire journey is 80 kmph. Find x.

Again since the distances are equal, using the formula, we have

$$\frac{2 \times 30 \times x}{30 + x} = 80 \Rightarrow 60x = 2400 + 80x \text{ i.e. } x = -120 \text{ ???}$$

Obviously something is wrong some place, but then the formula will not help us identify what's wrong.

Since the working is independent of the equal distances, let's assume the distances to be 240 kms (LCM of 30 and 80) to avoid calculations and also to realize what is going wrong.

To travel the first 240 km at 30 kmph, one would take $\frac{240}{30}$ = 8 hours.

Since average speed of the entire journey is given, the time take for the entire journey is $\frac{480}{80}$ = 6 hours. How can this be possible? This means the second stretch of 240 kms should be covered in –2 hours! Hence we are getting the weird answer. The working would be similar whatever value of the distance you take.

What this means is that if the first stretch of two equal distances is run at s kmph, the average speed for the entire journey in no case can be equal to $2s$ kmph or more than that. Why?

Some time is taken for first stretch of two equal stretches. Over entire journey, distance has doubled, and for average speed also to double, the second stretch has to be covered in 0 time. Even if one travels the second stretch in 1 sec or lesser than that, yet one would have covered the entire journey (twice the distance) in little more than the time taken on first stretch, so the average speed cannot be twice or more than that.

So in the above example, the data given is inconsistent and an average speed of 80 kmph is not possible with first stretch of equal distance being run at 30 kmph.

Exercise

37. A car travels the first one-third of a certain distance with a speed of 10 kmph, the next one-third of the distance with a speed of 20 kmph and the last one-third distance at 60 kmph. Find the average speed of the car for the whole journey.

1. 18 kmph 2. 20 kmph 3. 25 kmph 4. 30 kmph 5. 40 kmph

38. A car travels the first one-third of the total time taken to cover a distance at a speed of 10 kmph, the next one-third of time at a speed of 20 kmph and the last one-third time at 60 kmph. Find the average speed of the car for the whole journey.

1. 18 kmph 2. 20 kmph 3. 25 kmph 4. 30 kmph 5. 40 kmph

39. A person travels from P to Q at a speed of 40 kmph. By what percent should he increase his speed on the return journey so that his average speed for the round trip is 48 kmph?

1. 40 % 2. 50 % 3. 60 % 4. 75 % 5. 80 %

40. The average speed of a bus, excluding all stoppage time is 54 kmph and including all stoppage time is 45 kmph. Find the duration (in minutes) that the bus stops every hour.

1. 6 min 2. 10 min 3. 12 min 4. 15 min 5. 20 min

41. A car first travels uphill at a speed of 54 kmph and then downhill at a speed of 72 kmph such that the average speed for the entire journey is 63 kmph. Find the ratio of the uphill distance to the downhill distance.

1. 1 : 1 2. 6 : 7 3. 7 : 8 4. 3 : 4 5. Cannot be determined

Circular Motion

These types of problems deal with athletes running on a circular track and questions being asked about when, where and how often would two or more athletes cross each other.

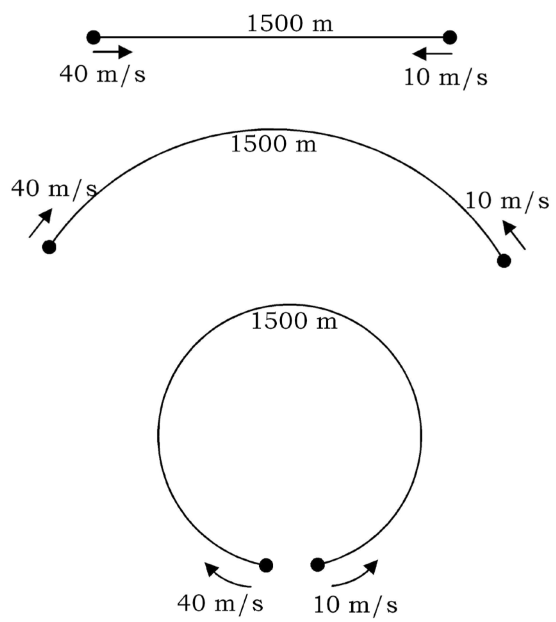
Meeting for the First Time

When running in opposite directions:

Consider two friends, A and B, who are separated by 1500 meters and they move towards each other at speeds of 40 m/s and 10 m/s. After how much time would they meet?

This is a straight question on relative speed and the time taken to meet = $\frac{1500}{40+10}$ = 30 sec.

Now instead of they traveling in a straight line, consider that the road they are traveling on is curved as shown. Would it make any difference to the time taken for them to meet? See the figure ...



If you answered – No, you are right. Even now A would cover 40 meters each second and B would cover 10 meters each second, so they would come closer by 50 meters each second.

If that is the case, if the road was even more curved i.e. almost circular as shown, how much time would they take to meet? (They cannot turn around and meet each other, they have to travel along the entire circular track to meet).

In this case also they would take the same time i.e. $\frac{1500}{40+10}$ = 30 sec

Thus time taken to meet = $\frac{\text{Track Length}}{\text{Relative Speed}}$

When running in the same direction:

Consider a police traveling at 40 m/s chasing a thief traveling at 10 m/s. If the distance between the police and thief when the chase starts is 1500 meters, find the time after which the police catches up with the thief.

Again, the solution should be known by now as $\frac{1500}{40-10}$ = 50 sec

What one should note in this problem and solution is that when the police catches-up with the thief, the police man would have run 1500 meters *more than* the thief i.e. the police man has to cover / make-up a distance of 1500 meters over the thief. The same situation exists when two athletes are running on a circular track in the same direction. An instant after the start, looking from the faster one's perspective, he sees a distance equal to the track length that is separating him and the slower one. To catch up with the slower one, he has to cover an entire track length *more than* the slower one. Thus even when the two athletes are running in the same direction, the time taken to meet = $\frac{\text{Track Length}}{\text{Relative Speed}}$. In this case the relative speed will be the difference in speeds.

Difference between Linear & Circular

Running on a circular track was explained above by drawing an analogy with meeting/catching up when two persons are running on a linear track. And we saw that the time taken to meet is found using the same formula, $\frac{\text{Distance seperating them}}{\text{Relative Speed}}$, except that the distance now is more appropriately the Track Length.

So then what is the difference between the two case? As long as one is interested only in the first meeting, there is essentially no difference. It is just that in the case of circular tracks if the two keep running after the first meeting, they will meet again and yet again and yet again. This does not happen in the case of a linear track. So tougher questions on circular tracks would not be about first meeting but about further meetings.

Frequency of meeting

Say two athletes start running from a same point on a circular track and meet after t units of time. When they meet they are again together at a point (not necessarily the same point where they started but nevertheless they are together). And if they continue running in the direction they were running and at their respective speeds, then this instance can be thought of as a new beginning, two athlete starting to run from a same point. And thus, from this instant onwards, they would again meet after t units of time. And the reasoning could be again argued similarly at their 2nd meeting. So considering the 2nd meeting as a fresh start, they will again meet after t duration of time. Thus, the time after which they meet for the first time is also the frequency with which they keep meet if they continue running at their respective speeds and respective directions.

If the first meeting takes place after t duration after start, the n^{th} meeting will take place after $n \times t$ duration after start.

Where would the first meeting take place?

Once the time is known after which two athletes, running on a circular track, meet, we can find the distance run individually by them (rather by any one of them) and find the exact position on the track where the meeting takes place. Rather than measuring this distance in absolute terms, it is more beneficial to measure this distance in terms of the track length.

In the above example, when running in opposite direction, the two athletes meet after 30 sec. In 30 sec, A would have run 40 × 30 = 1200 m. In terms of the track length this will be $\frac{1200}{1500} = \frac{4}{5}^{th}$ of the track length. It would be quite obvious that B in the same time would have run $\frac{1}{5}^{th}$ of the track length. And since they are in opposite directions both of them will be at the same point because measuring $4/5^{th}$ in one direction and $1/5^{th}$ in other direction will lead to the same point.

And when running in same direction, they meet after 50 secs and in this time A would have run 40 × 50 = 2000 m. This in terms of track length will be $\frac{2000}{1500} = \frac{4}{3}^{th}$ i.e. $1\frac{1}{3}^{rd}$ of the track length. B in the same time, would have run 10 × 50 = 500 i.e. $\frac{1}{3}^{rd}$ the track length. And since they are running in the same direction, they both are at the same point i.e. $\frac{1}{3}^{rd}$ of the track length, from the starting point in the direction they are running.

The following are simple questions based on the above, but will make you familiar and conversant with the approach. Also they will lead you on to an alternative approach of understanding the scenario of running on circular tracks. This alternative approach is explained in the box after example 34

Directions for E.g. 31 to 34: Three athletes A, B and C are running on a circular track of length 1200 meters with speeds 30 m/s, 50 m/s and 80 m/s. A is running clockwise and B and C are running anticlockwise.

E.g. 31: Find the time after which A and B will meet for the first time and the frequency (in seconds) after which they will keep meeting. Also find the sum of the distance (in fraction of the track length) run by them till their first meeting.

Since A and B are running in opposite direction, they would meet after $\frac{1200}{30+50} = \frac{1200}{80}$ = 15 sec.

From the first meeting onwards, one could say that A and B are again starting simultaneously afresh and running in opposite direction. Thus they would meet again after further 15 seconds. Continuing the same argument, we can say that A and B meet after every 15 seconds.

In 15 seconds, A would have run 15 × 30 = 450 meters i.e. $\frac{450}{1200} = \frac{3}{8}^{th}$ of the track length.

In 15 seconds, B would have run 15 × 50 = 750 meters i.e. $\frac{750}{1200} = \frac{5}{8}^{th}$ of the track length.

Notice that, when running in opposite directions, on meeting for the <u>first</u> time, they have <u>*together*</u> run <u>one</u> full track length.

E.g. 32: Find the first time that A and B will meet at the starting point. Also find which numbered meeting will this be. Find the sum of the distance (in terms of the track length) that they have run so far.

Meeting for the First Time <u>**at Start**</u>

The question of identifying after how much time would two or more athletes meet *at the starting point* is an application of LCM rather than a problem of Time Speed Distance. The key word here is meeting *at the starting point*. Please realize that this may not be the first time that they meet. The could have crossed each other (met) at some other point on the track but then that would not be counted as a meeting point for this question as it has not occurred at the starting point.

Consider two athletes, A and B running on a circular track and taking x and y units of time to complete one full circle.

A would reach the starting point for the first time after x units and thereafter would be at the starting point after every x units. Similarly B would reach the starting point for the first time after y units and thereafter would be at the starting point after every y units.

Thus, A and B would both be at the starting time after common multiple of x and y and the first time that this would occur would be the LCM of x and y.

For these types of problems, it does not matter in which direction the two athletes are running. Even if both are running clockwise or if one is running clockwise and other anticlockwise, the time when they would be at the starting point would remain the same.

A would be at the start after every multiple of $\frac{1200}{30}$ = 40 sec and B will be at the start after every $\frac{1200}{50}$ = 24 sec. Thus they would be simultaneously at the starting point for the first time at the LCM of 40 and 24 i.e. 120 seconds.

To find which numbered meeting will this be: In the above example we have found that A and B meet for the first time after 15 seconds and would keep meeting after every 15 seconds thereafter. Thus the meeting after 120 seconds will be their $\frac{120}{15}$ i.e. 8th meeting.

The distance run by A so far = 120 sec × 30 m/s = 3600 meters i.e. 3 full rounds.

The distance run by B so far = 120 sec × 50 m/s = 6000 meters i.e. 5 full rounds.

Thus, *together* they have run eight full rounds and this is their eighth meeting. Remember they are running in opposite direction.

From these two solved examples, one should be clear that when two athletes are running in opposite direction, they meet whenever *together* they have run one full round.

E.g. 33: Find the time after which B and C will meet for the first time and the frequency (in seconds) after which they will keep meeting. Also find the difference of the distance (in fraction of the track length) run by them till their first meeting.

Since B and C are running in same direction, they would meet after $\frac{1200}{80-50} = \frac{1200}{30}$ = 40 sec.

From the first meeting onwards, one could say that B and C are again starting simultaneously afresh and running in same direction. Thus they would meet again after further 40 seconds. Continuing the same argument, we can say that B and C meet after every 40 seconds.

In 40 seconds, B would have run 40 × 50 = 2000 meters i.e. $\frac{2000}{1200} = \frac{5}{3}^{th}$ i.e. $1\frac{2}{3}^{rd}$ of the track length.

In 40 seconds, C would have run 40 × 80 = 3200 meters i.e. $\frac{3200}{1200} = \frac{8}{3}^{th}$ i.e. $2\frac{2}{3}^{rd}$ of the track length.

Notice that, when running in same direction, on meeting for the first time, the faster athlete has run one full track length *more* than the slower.

E.g. 34: Find the first time that B and C will meet at the starting point. Also find which numbered meeting will this be. Find the difference of the distance (in terms of the track length) that they have run so far.

B would be at the start after every multiple of $\frac{1200}{50}$ =24 sec and C will be at the start after every $\frac{1200}{80}$ = 15 sec. Thus they would be simultaneously at the starting point for the first time at the LCM of 24 and 15 i.e. 120 seconds.

To find which numbered meeting will this be: In the above example we have found that B and C meet for the first time after 40 seconds and would keep meeting after every 40 seconds thereafter. Thus the meeting after 120 seconds will be their 3rd meeting.

The distance run by B so far = 120 sec × 50 m/s = 6000 meters i.e. 5 rounds

The distance run by C so far = 120 sec × 80 m/s = 9600 meters i.e. 8 rounds

The difference of the distance run by them is 3 rounds i.e. the faster athlete has run <u>three</u> full rounds <u>*more*</u> than the slower one and this is their <u>third</u> meeting.

From these two solved examples, one should be clear that when two athletes are running in same direction, whenever the faster one has run one full round *more* than the slower one they would meet.

Another Important Approach: 'Relative' Distances or Rounds run

When running in opposite directions ......

...... they meet whenever *together* they have covered one full round ......

...... and hence at the n^{th} meeting, *together* they would have covered n rounds.

This is easy to understand. Since both start from common point and run in opposite directions, at 1st meeting, if one runs f fraction of the track, the other has to run $(1 - f)$ fraction of the track so that he is at same point. Thus, sum of distances run = $f + (1 - f)$ i.e. 1 full round.

This is new beginning, and from now onwards, till 2nd meeting they would together again cover 1 full round i.e. they would together cover 2 rounds since start. And so on.

When running in same directions ......

...... they meet whenever the faster one has covered 1 round more than the slower one ......

...... and hence at the n^{th} meeting, the faster one would have covered n rounds more than the slower one.

Consider both start from same point and run in same direction. Focus on the gap between them. The faster one will race ahead of the slower one and a gap will start emerging between them. As time passes the gap will start increasing.

...... cont'd

Slowing the gap will increase from being $\frac{1}{10}^{th}$ of the track to $\frac{1}{5}^{th}$ of the track and after some more time, the gap will increase to $\frac{1}{2}$ of the track (the gap increases linearly, the values chosen are random ones to explain). When the gap is $\frac{1}{2}$, they will appear diametrically opposite. Note that so far they would not have met (and we also have no idea how many circles each would have done individually so far, could be any number – fraction of the track or many rounds of the tracks). As they keep running, the gap increases from $\frac{1}{2}$ to $\frac{3}{4}^{th}$ of the track length. (For an observer, the slower one will appear to be ahead of the faster one). And after some more time, the gap will be exactly one full round. When the gap is exactly one full round, the faster one will be at a point that is 1 round ahead of the point where the slower one will be. Both these points will be the same i.e. they will be together. From now again the gap again starts increasing, linearly, increasing to $1\frac{1}{10}$, then to $1\frac{1}{5}$, then to $1\frac{1}{2}$, then to $1\frac{3}{4}$ and finally to 2. It is only at this moment that the faster one would again be together with the slower one i.e. this will be the 2nd meeting.

The above logical way of looking at circular tracks can be used very effectively in solving questions that go beyond the first meeting.

Get an idea of the number of rounds covered by them, say a and b rounds in the same time (mind you, a and b could also be fractions). If they are running in the same direction, the number of times they meet is $[a - b]$ and if they are running in the opposite direction, the number of times they meet is $[a + b]$, where $[x]$ refer to the integral part of x. This logic can also be used in the reverse way as explained in e.g. 37

E.g. 35: In a race of 4800 meters run on a circular track of 400 meters length, the ratio of the speed of the two athletes is 3 : 5. If they run in the same direction, how many times do they meet in the entire race?

Approach 1: Short-cut

Since it is a 4800 meter race on a track of 400 meters, the winner would have to complete $\frac{4800}{400}$ = 12 laps or rounds. Since the ratio of the speeds is 3 : 5, when the faster guy completes 12 round, the slower guy would have done $\frac{3}{5} \times 12 = \frac{36}{5} = 7\frac{1}{5}$ or 7.2 rounds. The faster one has run 12 – 7.2 = 4.8 rounds more than the slower one and hence would have met him four times.

(The difference in the number of rounds is not an integer means that when the race ends the two athletes are not together, the faster athlete is 4.8 rounds ahead of the slower. The last meeting was when the faster one was 4 rounds ahead and since then he has gone ahead of the slower guy and the race ends when the faster one is 0.8 i.e. of the track ahead of the slower one. But since the race ends here, he does not overtake (meet) the slower guy for the fifth time.)

Approach 2: Multiple of 1st meeting

If the speed of the athletes is assumed as $3k$ and $5k$, then the two athletes would meet every $\frac{400}{5k-3k}=\frac{200}{k}$ secs.

And the race would last for $\frac{4800}{5k}=\frac{960}{k}$ secs.

If meeting occurs every $\frac{200}{k}$ secs, the number of times they would meet in $\frac{960}{k}$ secs is $\frac{960\cancel{/k}}{200\cancel{/k}}$ = 4.8 i.e. 4 times.

E.g. 36: In the above race, had they been running in opposite directions, how many times would they have met?

When the faster one completes 12 rounds, the slower one completes 7.2 rounds. Thus they have together done 12 + 7.2 = 19.2 rounds. So they would have met 19 times.

E.g. 37: A and B can complete one full round of the circular jogging track in 15 and 40 minutes respectively. If they start at the same time from the same point and run in the same direction, after how much time will they meet for the first time?

Approach 1: Short-cut

Since the ratio of time taken for 1 round, same distance, is 3 : 8, the ratio of their speeds will be 8 : 3. Thus, in the time A runs 8 rounds, B runs 3 rounds i.e. A runs 5 rounds more than B. Dividing this by 5, in the time A runs $\frac{8}{5}$ rounds, B will run $\frac{3}{5}$ rounds i.e. A runs 1 round more than B i.e. this is their 1st meeting. If A takes 15 minutes to run 1 1round, to cover $\frac{8}{5}$ rounds he will take $\frac{8}{5}\times 15$ = 24 minutes.

Approach 2:

One could also have solved like the traditional way by assuming the track length to be d. Thus, the speeds of A and B are $\frac{d}{15}$ and $\frac{d}{40}$. And the time for 1st meeting will be $\frac{d}{\frac{d}{15}-\frac{d}{40}}=\frac{15\times 40}{40-15}=\frac{15\times 40}{25}$ = 3 × 8 = 24.

E.g. 38: A runs in the clockwise direction on a circular track whereas B runs on the same track but in the anti-clockwise direction, but they start from the same point. If their speeds are 25 m/s and 40 m/s respectively, at what fraction of the track-length, measured in clockwise direction from the starting point, does the 10^{th} meeting take place?

Ratio of speeds is 5 : 8.

Thus in the time A runs 5 rounds, B will run 8 rounds.

Dividing by 5 + 8 = 13, in the time A runs $\frac{5}{13}$ rounds, B will run $\frac{8}{13}$ rounds and this will be their 1^{st} meeting.

Multiplying this by 10, at the time of the 10^{th} meeting A will have run $\frac{50}{13} = 3\frac{11}{13}$ rounds i.e. the meeting place, measured in the clockwise direction, is $\frac{11}{13}^{th}$ of the track length.

E.g. 39: If A, B and C take 12 mins, 15 mins and 20 mins to complete one full round of a circular track, after how much time will all three A, B and C meet for the first time, if they start from the same point at the same time, with A and C running in clockwise direction and B running in anti-clockwise direction?

> Three or more people meeting
>
> To find time when three or more people meet, find the time after which pairs of athletes meet, such that at least one athlete is common to all the pairs (say, A & B, A & C, A & D, ...). The respective pairs will keep meeting after any multiple of the time found. At the LCM of these durations of time, we are sure that A & B have met, that A and C have met, that A & D have met and so on. But the only way that A could have met all these people at the same instant is when all of them are together.

Considering A & B: Taking the LCM of their time, in 60 minutes, A runs 5 rounds and B will run 4 rounds. Since they are running in opposite directions, they would have met 5 + 4 = 9 times. Thus, in $\frac{60}{9} = \frac{20}{3}$ mins, they would meet for the first time and will meet every $\frac{20}{3}$ mins.

Considering A & C: Taking the LCM of their time, in 60 minutes, A runs 5 rounds and C will run 3 rounds. Since they are running in same directions, they would have met 5 – 3 = 2 times. Thus, in $\frac{60}{2}$ = 30 mins, they would meet for the first time and will meet every 30 mins.

All three would meet after time equal to the LCM of $\frac{20}{3}$ & 30.

Using LCM of fraction = $\frac{\text{LCM of numerators}}{\text{HCF of denominators}}$, the required answer is

$\frac{\text{LCM }(20,30)}{\text{HCF }(3,1)} = \frac{60}{1}$ = 60 sec.

Position of the meeting points

These types of questions do not deal with when or how often do the athletes meet when they are running on a circular track. These questions pertain to the number of points and their placement on the circular track where the athletes can possibly meet.

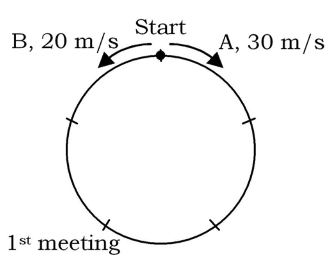
When running in the opposite directions:

If the ratio of speeds of two athletes (in the most reducible form) is $a : b$, the number of distinct meeting points on the track would be would be $a + b$. This funda is best understood with an example...

Consider athletes A and B running at speeds of 30 m/s and 20 m/s on a circular track of 1000 meters, A running clockwise and B anti-clockwise. If they keep running indefinitely, at how many distinct point on the circle would they meet?

A and B would meet for the first time after $\frac{1000}{30+20} = \frac{1000}{50}$ = 20 sec.

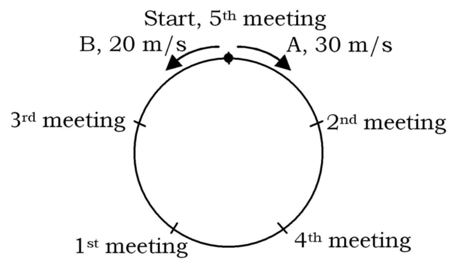
In 20 seconds, A would travel 30 × 20 = 600 i.e. $\frac{600}{1000} = \frac{3}{5}^{th}$ of the circle. To find this point precisely, it is best to divide the circle in 5 equal parts. Thus they would meet at the point as shown in the figure:



Compared to the starting point, the meeting point is 3 segments clockwise.

Now this point could be considered as a fresh start and thus, they would again meet after A has run $\frac{3}{5}^{th}$ of the circle again i.e. 3 segments clockwise from the point of first meeting.

This logic can be extended and the following figure shows the 2nd, 3rd, 4th and 5th meeting points.



We see that the 5th meeting happens at the starting point. This is not just 'can be considered as a fresh start' but is exactly same as the start since not only are they together but they are also at the same point from where they started. Thus after this the same pattern will start repeating and even if the runners keep running indefinitely they would just meet at these 5 points.

In short, if the ratio of two runners running in opposite directions, in the most reducible form is $a : b$, they can meet at $a + b$ distinct points on the track.

When running in the same direction:

If the ratio of speeds of two athletes (in the most reducible form) is $a : b$, the number of distinct meeting points on the track would be would be $|a - b|$.

Consider a similar example as above but with different values: A and B are running with speeds 40 m/s and 10 m/s on a circular track of length 1500 meters, but this time they are running in the same direction, clockwise.

They would meet for the first time after $\frac{1500}{40-10}$ = 50 sec.

In 50 seconds, A would have run 40 × 50 = 2000 meters i.e. $\frac{2000}{1500} = \frac{4}{3}^{th} = 1\frac{1}{3}^{rd}$ of the circle and B would have run 10 × 50 = 500 i.e. $\frac{500}{1500} = \frac{1}{3}^{rd}$ of the circle. As expected, B has run 1 round more than A and they have met. And this meeting point is $\frac{1}{3}^{rd}$ of a circle in the clockwise direction from start. Considering this as a new start, they should meet at $\frac{1}{3}^{rd}$ of a circle ahead (clockwise) from this point i.e. at a point $\frac{2}{3}^{rd}$ clockwise from the starting point. Considering this as a yet new start, they should meet at $\frac{1}{3}^{rd}$ of a circle ahead (clockwise) from this point i.e. at the start. And since this is exactly similar to the original start, even if they keep running indefinitely, they would just keep meeting at these 3 points.

If the ratio of two runners running in the same direction, in the most reducible form is $a : b$, they can meet at $|a - b|$ distinct points on the track.

And in both the cases, opposite directions and same direction, irrespective of the number of points, one of the poitn will be the starting point and the others will be equally spaced out on the track. Thus, three points of meeting would mean the three points will be the starting point and points at 1/3rd and 2/3rd from starting point.

E.g. 40: Which of the following cannot be the ratio of speeds of two joggers running on a circular jogging track if while running they meet at a diametrically opposite point to the point from where both of them started?

(1) 3 : 5 (2) 1 : 3 (3) 1 : 5 (4) 2 : 5

Evaluating each option ...

Option (1): With ratio of speeds being 3 : 5, if they are running in same directions, then they would meet at 5 – 3 = 2 points, the starting point and the diametrically opposite point.

Option (2): With ratio of speeds being 1 : 3, if they are running in same directions, then they would meet at 3 – 1 = 2 points, the starting point and the diametrically opposite point.

Option (3): With ratio of speeds being 1 : 5, if they are running in same directions, then they would meet at 5 – 1 = 4 points, of which one will be the diametrically opposite point to the point from where they started.

Option (4): With ratio of speeds being 2 : 5, if they are running in same directions, then they would meet at 5 – 2 = 3 points. The 3 points will be the starting point, point that is $1/3^{rd}$ of the track length from the starting point and point that is $1/3^{rd}$ of the track length from the starting point. Thus, they would not meet at the diametrically point if this is their ratio of speeds.

Even if they were running in opposite direction, they would meet at 7 points, the starting point and points at distance of $\frac{1}{7}^{th}, \frac{2}{7}^{th}, \frac{3}{7}^{th}, ..., \frac{6}{7}^{th}$ of track length. Even in this case, the diametrically opposite point will not be a meeting point.

Exercise

42. A and B can run one full round of a circular track in 8 minutes and 15 minutes respectively. If they start simultaneously and run in same direction, after how much time will they meet for the first time?

1. 24 mins 2. $\frac{120}{7}$ mins 3. $\frac{120}{17}$ mins 4. $\frac{120}{23}$ mins 5. 120 mins

43. Arjun and Bhim can run a full round around a circular track in 4 minutes and 7 minute respectively. If they start simultaneously, after how much time will they meet together at a point diametrically opposite form the starting point?

1. 3 mins 2. 5.5 mins 3. 11 mins 4. 28 mins 5. Never

44. A, B and C can complete one full round of a circular track in 10 minutes, 30 minutes and 50 minutes respectively. If A runs clockwise and B and C run anticlockwise, find the time after which they will meet for the first time.

1. 20 mins 2. 30 mins 3. 60 mins 4. 75 mins 5. 150 mins

45. What should be the length of the race, if it is run on a track of 400 meters and it should end exactly on the 5^{th} meeting of A and B, both of who run clockwise with speeds of 35 m/s and 45 m/s.

1. 7 km 2. 8 km 3. 9 km 4. 10 km 5. 12 km

46. Four athletes run a race, starting from the same point and all of them running clockwise. If the ratio of the speeds of the athletes are 1 : 2 : 3 : 4, at how many distinct points on the circular track could any two athletes meet (overtake)?

1. 4 2. 5 3. 6 4. 8 5. 10

47. A and B start running on a circular track of length 1200 meters with speeds in the ratio of 1 : 5 and A running clockwise and B anticlockwise. At what distance from the starting point, measured clockwise, would the point of their 4^{th} meeting be?

1. 200 mts 2. 400 mts 3. 600 mts 4. 800 mts 5. 1000 mts

48. In a race between A and B, both start simultaneously from the same point but A runs clockwise and B runs anticlockwise. They meet for the first time at a distance of 300 meters clockwise from the starting point and for the second time at a distance of 200 meters anticlockwise from the starting point. Find the ratio of speeds of A and B, if it is known that A has not completed one full round until the second meeting.

    1. 3 : 2 2. 1 : 1 3. 3 : 5 4. 1 : 3 5. Cannot be determined

49. In the above question, if A had already completed one round before the second meeting, find the length of the track.

    1. 400 mts 2. 500 mts 3. 600 mts 4. 800 mts 5. 1000 mts

50. In a race of 4800 meters run on a circular track of 400 meters length, the ratio of the speed of the two athletes is 3 : 5. If they run in the same direction, how many times do they meet in the entire race?

    1. 4 2. 5 3. 6 4. 7 5. 8

51. A and B run a race, in the same direction on a circular track of 100 meters. If the race was of 2000 meters and the ratio of speeds of A and B was 5 : 3, find the number of times the two meet?

    1. 4 2. 5 3. 6 4. 7 5. 8

52. A and B start a race of 10 laps in a swimming pool, from the same end. One lap is going to and fro the entire length of the pool. If speed of A and B is 4 m/s and 5 m/s, where would their 9th meeting be?

    1. At starting end

    2. At mid-point of length of pool

    3. At the opposite end of starting end

    4. They would never meet for the 9th time.

    5. Depends on length of the pool

53. At how many distinct points in a watch do the hour and minute hand meet?

    1. 10 2. 11 3. 12 4. 13 5. Infinite

Time & Work

Questions on work usually fall in one of the following categories:

1. Relation between days taken by individuals to complete a given work independently and to complete while working simultaneously or alternately.

2. Teams of men, women, children and time taken by the teams to complete a work independently or working simultaneously.

Most questions asked are standard types of questions. Also the approach to solve questions is very standard one. There are two approaches, which are exactly the same but seem to be different – approach using the per day's work and approach using LCM. The two approaches are best explained through an example.

If *A* does a work in 10 days and *B* does the same work individually in 12 days, in how many days will the work be completed if they work simultaneously?

Approach 1: Per day's work

Basics

If A takes x days to do a piece of work, then each day he would be doing $\frac{1}{x}^{th}$ of the work.

Conversely, if in one day f fraction of work is done, then complete work will be done in $\frac{1}{f}$ days.

Since A completes the entire work in 10 days, A does $\frac{1}{10}^{th}$ of the work in 1 day.

Since B completes the entire work in 12 days, B does $\frac{1}{12}^{th}$ of the work in 1 day.

Working simultaneously, they do $\frac{1}{10}+\frac{1}{12}=\frac{11}{60}^{th}$ of the work in 1 day

Thus total days taken by both working simultaneously = $\frac{60}{11}$ days

Same as Time Speed & Distance

This topic in a lot of books is covered under the topic of time, speed and distance. There are a lot of similarities in the above solution to that of time, speed and distance.

Covering a distance in certain time is similar to completing a work in a certain time. Thus, the rate at which work is done is the equivalent of speed.

Approach 2: LCM

Let the amount of work be 60 units (LCM of 10 and 12)

Since *A* does 60 units in 10 days, he does 6 units every day

Since *B* does 60 units in 12 days, he does 5 units every day.

Working simultaneously, they do 6 + 5 = 11 units each day.

Thus to complete 60 units of work, they will take $\frac{60}{11}$ days.

The two approaches are absolutely identical, it is just that in the earlier approach the work was assumed as 1 unit instead of 60 units.

The above example illustrates all the theory that you will have to use. All other matter in a question is building a story around the above.

E.g. 1: If *A* does a work in 10 days and *B* does the same work individually in 12 days. *A* starts working alone and after 3 days is joined by *B*. After another 1 days, *A* quits. How many more days will be taken by *B* to complete the work?

Approach 1: Per day's work

A is effectively working for 4 day and thus will do $\frac{4}{10}$ i.e. $\frac{2}{5}^{th}$ of the work.

B has to do $\frac{3}{5}^{th}$ of the work and he will take a total of $\frac{3/5}{1/12} = \frac{36}{5} = 7.2$ days for it. He has already put in 1 day of work. So he will take another 6.2 days

You should also be familiar with the equation: $\frac{4}{10} + \frac{d}{12} = 1$, where d is the days *B* has worked. The right hand side of the equation is 1 because the entire work is done and in this approach the entire work is assumed as 1. This equation will also give us d = 7.2

Approach 2: LCM

Let amount of work be 60 units.

Thus *A* does 6 units per day and *B* does 5 units per day.

A works for total of 4 days and competes 24 units. *B* does work for 1 day and does 5 units. Thus 29 units of the work is already done and 31 units are remaining which will be done at the rate of 5 units per day. Thus, $\frac{31}{5}$ = 6.2 more days will be taken.

Concept of Negative Work, Pipes and Cisterns

While in the above examples, both players were working to get the work done, it is also quite possible that one of them might be working towards 'breaking' the work i.e. doing negative work. Such type of problems do appear in the above contexts also, but are more popular in questions called 'Pipes and Cisterns'. These questions are exactly the same as above where the work to be done is to fill a cistern (anything that collects water e.g. tank, bucket, etc) using inlet pipes. In case of cistern, a leak at the bottom of the tank or an outlet pipes denotes negative work.

E.g. 2: An inlet pipe can fill in an empty cistern in 30 minutes whereas a leak in the bottom of the cistern can empty a filled tank in 40 minutes. Find the time taken to fill the cistern when both the inlet pipe and the leak are on.

Part of cistern that is filled each minute = $\frac{1}{30} - \frac{1}{40} = \frac{1}{120}$. Thus entire cistern is filled in 120 minutes.

E.g. 3: Two inlets pipes can individually fill a cistern in 10 minutes and 20 minutes respectively. However it took them 8 minutes to fill the cistern working simultaneously, because of a leak at the bottom of the cistern. In how many minutes would the leak empty the entire filled cistern if the inlet pipes are closed?

Since cistern is filled in 8 minutes, each minute $\frac{1}{8}^{th}$ of it is getting filled. Thus if the leak can empty the filled cistern in x minutes, we have

$$\frac{1}{10} + \frac{1}{20} - \frac{1}{x} = \frac{1}{8} \Rightarrow \frac{1}{x} = \frac{4+2-5}{40} = \frac{1}{40}.$$

Thus the leak can empty the cistern in 40 minutes.

Teams of men, women and children

E.g. 4: If 4 boys or 3 men can complete a piece of work in 10 days how many days will it take for 4 men and 3 boys to finish the work?

Approach 1: Converting to one denomination only.

The team of 4 boys is equivalent to team of 3 men, since both these team takes the same number of days. Thus, 1 boy is equivalent to $\frac{3}{4}$ men.

And hence 3 boys will be equivalent to $\frac{9}{4}$ men. Now the combination of 4 men and 3 boys can be thought of as $4 + \frac{9}{4} = \frac{25}{4}$ men. 3 men complete the work in 10 days. Hence 25/4 men can complete the work in

$$\underbrace{\frac{3}{25/4}}_{\text{more men} \Rightarrow \text{less days}} \times 10 = \frac{3 \times 4 \times 10}{25} = \frac{24}{5} \text{ days.}$$

Approach 2: Treating them as two individuals working simultaneously.

If 4 boys can finish the work in 10 days, then 3 boys can finish it in

$\underbrace{\frac{4}{3}}_{\text{less boys} \Rightarrow \text{more days}} \times 10 = \frac{40}{3}$ days. Thus, 3 boys do $\frac{3}{40}^{th}$ of the work in 1 day

If 3 men can finish the work in 10 days, then 4 men can finish it in

$\underbrace{\frac{3}{4}}_{\text{more men} \Rightarrow \text{less days}} \times 10 = \frac{15}{2}$ days. Thus, 4 men do $\frac{2}{15}^{th}$ of the work in 1 day.

Thus work done by 3 boys & 4 men in 1 day = $\frac{3}{40} + \frac{2}{15}$ i.e. $\frac{25}{120}$ i.e. $\frac{5}{24}^{th}$

of work. Thus days taken to complete the work = $\frac{24}{5}$ days.

E.g. 5: 1 man and 2 women can complete a work in 10 days. 2 men and 3 women can complete the work in 6 days. Find the number of days needed by 4 men and 4 women to complete the work.

Approach 1: Treating them as two individuals working simultaneously

If 1 man, working alone, takes *m* days to finish the entire work and 1 woman, working alone, takes *w* days to finish the entire work, then,

$\frac{1}{m} + \frac{2}{w} = \frac{1}{10}$ ...... (i) $\frac{2}{m} + \frac{3}{w} = \frac{1}{6}$ ...... (ii)

2 × (i) – (ii) $\frac{1}{w} = \frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Substituting $1/w$ in (i), $\frac{1}{m} = \frac{1}{10} - \frac{1}{15} = \frac{1}{30}$

Work done by 4 men and 4 women in 1 day will be $\frac{4}{m} + \frac{4}{w} = \frac{4}{15}^{th}$ of the work. Thus, the number of days taken is $\frac{15}{4}$ days.

Approach 2: Converting to one denomination only

1 man and 2 women can complete the work in 10 days. Hence if work has to be completed in 1 day, we would need 10 such teams i.e. 10 men and 20 women.

2 men and 3 women can complete the work in 6 days. Hence if work has to be completed in 1 day, we would need 6 such teams i.e. 12 men and 18 women.

Thus (10 men & 20 women) is equivalent to (12 men & 18 women) i.e. 2 women = 2 men i.e. one woman is equivalent to one man.
Thus, 3 persons can complete the work in 10 days and hence 8 individuals can complete the work in $\underbrace{\frac{3}{8}}_{\text{more men} \Rightarrow \text{less days}} \times 10 = \frac{15}{4}$ days.

Exercise

1. *A* tank, which usually takes 5 hours to be filled, takes 6 hours to fill because of a leak. Find the time in which the leak can empty the tank if it is filled and the inlet pipe is shut.

   1. 30hrs   2. 27.5hrs   3. 25hrs   4. 22.5hrs   5. 20hrs

2. Three taps *A*, *B* and *C* can fill a tank individually in 10 hrs, 20 hrs and 25 hrs respectively. At first all the taps are opened simultaneously. After 2 hours tap *C* is closed and after further 2 hours tap *B* is also closed. Tap *A* is kept open till the tank gets completely filled. What fraction of the tank is filled by tap *A*?

   1. $\frac{2}{11}$ 2. $\frac{9}{11}$ 3. $\frac{18}{25}$ 4. $\frac{7}{25}$ 5. $\frac{1}{2}$

3. 4 men and 3 women can finish a piece of work in 6 days whereas if there were 5 men and 7 women, they would have taken 4 days to finish the work. How many days will 1 man and 1 woman take to finish the work?

   1. $\frac{156}{5}$ 2. $\frac{156}{7}$ 3. $\frac{156}{9}$ 4. $\frac{156}{11}$ 5. $\frac{156}{13}$

4. *A* and *B* can complete a work working individually in 60 days and 40 days respectively. Both start working simultaneously on the work simultaneously, but 4 days before the work is scheduled to get over, *B* leaves. Find the total number of days taken for the work to be completed.

   1. 24   2. 25   3. 26.4   4. 28.2   5. 30

5. *A* and *B* can complete a work working individually in 60 days and 40 days respectively. Both start working simultaneously on the work simultaneously, but *B* did not work on the last 4 days. Find the total number of days taken for the work to be completed.

   1. 24   2. 25   3. 26.4   4. 28.2   5. 30

6. *A* is thrice as efficient as *B* and hence takes 12 days less to complete a work as compared to *B*. Find the number of days in which they can complete the work while working together?

   1. 3   2. 4.5   3. 6   4. 6.5   5. 7.5

7. *A* is twice as efficient as *B* and *B* is thrice as efficient as *C*. If all three together can complete a work in 10 days, find the difference in the number of days taken to complete the work when *A* and *B* work together and when *B* and *C* work together.

   1. 25   2. $\frac{125}{6}$ 3. $\frac{125}{7}$ 4. $\frac{125}{8}$ 5. $\frac{125}{9}$

8. Tina can do as much work in 2 days as Leena can do in 3 days and Leena can do as much work in 4 days as Meena can do in 5 days. If all three work together, they take 20 days to finish a work. How long would Leena take to finish the work if she works alone?

   1. 50 days 2. 60 days 3. 66 days 4. 72 days 5. 75 days

9. *A* and *B* can finish a work in 10 days when working together. *B* and *C* working together can finish the same work in 12 days and *A* and *C* working together can finish the work in 15 days. In how many days will the work get over if all three *A*, *B* and *C* work together?

   1. 5 2. 6 3. 8 4. 4 5. 3

10. *A* and *B* completed a work together in 5 days. Had *A* worked at twice the speed and *B* at half the speed, it would have taken them only 4 days to finish the work. How much time would *A* take to finish the work if he worked alone?

    1. 5 2. 6 3. 8 4. 9 5. 10

11. *A* can build an entire wall in 50 days whereas *B*, who is a demolition man, can break the entire wall in 60 days. If they work on alternate days with *A* starting, in how many days will the wall be built completely for the first time?

    1. 299 2. 300 3. 599 4. 600 5. 589

12. The days taken to built a wall individually by each of *A*, *B*, *C*, *D* and *E* is 20 days, 15 days, 12 days, 10 days and 6 days. They divide themselves into two teams such that the days taken by the two teams to finish the work are in the ratio 1.5 : 4.5. Find the team that finishes earlier.

    1. A, B 2. C, D, E 3. B, C 4. A, D, E 5. B, E

13. Four small pumps and two large pumps are filling a tank. Each of the small pumps works at three-fifth the rate of the large pump. If all 6 pumps work at the same time, they should fill the tank in what fraction of the time that it would have taken if only one large pump and only one small pump were working together?

    1. 1/7 2. 2/9 3. 3/8 4. 4/11 5. 5/12

14. A can build a wall in 10 days, working alone, B can build the same wall in 20 days, working alone and C can break the entire built wall in 8 days, working alone. The three of them work alone on the wall on successive days with A working on first day, B on second day and C on third day and the cycle then repeats. In how many days will the wall be built for the first time?

    1. 102 days 2. 104 days 3. 112 days 4. 120 days 5. 124 days

15. Either of the team of A and B or team of C and D can do the a work in 15 days. Also the same work can be done by the team of A and D in 10 days and by the team of B and C in 30 days. If B alone takes 40 days to complete the work, find the ratio of the number of days taken by the team of B and D and that taken by the team of A and C to complete the work.

    1. 1 : 1 2. 3 : 5 3. 3 : 4 4. 1 : 2 5. Cannot be determined

CAT Questions

Note: The following question requires geometry awareness of 30-60-90 triangle.

1. *[CAT 2008]* Rahim plans to drive from city A to station C, at the speed of 70 km per hour, to catch a train arriving there from B. He must reach C at least 15 minutes before the arrival of the train. The train leaves B, located 500 km south of A, at 8:00 am and travels at a speed of 50 km per hour. It is known that C is located between west and northwest of B, with BC at 60° to AB. Also, C is located between south and southeast of A with AC at 30° to AB. The latest time by which Rahim must leave A and still catch the train is closest to

   (1) 6:15 am (2) 6:30 am (3) 6:45 am (4) 7:00 am (5) 7:15 am

Directions for Questions 2 & 3: *[CAT 2007]* Cities A and B are in different time zones. A is located 3000 km east of B. The table below describes the schedule of an airline operating non-stop flights between A and B. All the times indicated are local and on the same day.

Departure		Arrival	
City	Time	City	Time
B	8:00 am	A	3:00 pm
A	4:00 pm	B	8:00 pm

Assume that planes cruise at the same speed in both directions. However, the effective speed is influenced by a steady wind blowing from east to west at 50 km per hour.

2. What is the time difference between A and B?

   (1) 2 hours (2) 2.5 hours (3) 1 hour (4) 1.5 hours (5) Cannot be determined

3. What is the plane's cruising speed in km per hour?

   (1) 550 (2) 600 (3) 500 (4) 700 (5) Cannot be determined

4. *[CAT 2006]* Arun, Barun and Kiranmala start from the same place and travel in the same direction at speeds of 30 km/hr, 40 km/hr and 60 km/hr respectively. Barun starts two hours after Arun. If Barun and Kiranmala overtake Arun at the same instant, how many hours after Arun did Kiranmala start?

   (1) 3 (2) 3.5 (3) 4 (4) 4.5 (5) 5

Note: The following question requires basic geometry awareness.

5. *[CAT 2005]* A jogging park has two identical circular tracks touching each other, and a rectangular track enclosing the two circles. The edges of the rectangles are tangential to the circles. Two friends, A and B, start jogging simultaneously from the point where one of the circular tracks touches the smaller side of the rectangular track. A jogs along the rectangular track, while B jogs along the two circular tracks in a figure of eight. Approximately, how much faster than A does B have to run, so that they take the same time to return to their starting point?

   (1) 3.88% (2) 4.22% (3) 4.44% (4) 4.72%

Directions for questions 6 & 7: *[CAT 2005]* Ram and Shyam run a race between points A and B, 5 km apart. Ram starts at 9 a.m. from A at a speed of 5 km/hr, reaches B, and returns to A at the same speed. Shyam starts at 9:45 a.m. from A at a speed of 10 km/hr, reaches B and comes back to A at the same speed.

6. At what time do Ram and Shyam first meet each other?

   (1) 10 a.m. (2) 10:10 a.m. (3) 10:20 a.m. (4) 10:30 a.m.

7. At what time does Shyam overtake Ram?

   (1) 10:20 a.m. (2) 10:30 a.m. (3) 10:40 a.m. (4) 10:50 a.m.

8. *[CAT 2005]* A chemical plant has four tanks (A, B, C, and D), each containing 1000 litres of a chemical. The chemical is being pumped from one tank to another as follows:

   From A to B @ 20 litres/minute

   From C to A @ 90 litres/minute

   From A to D @ 10 litres/minute

   From C to D @ 50 litres/minute

   From B to C @ 100 litres/minute

   From D to B @ 110 litres/minute

   Which tank gets emptied first, and how long does it take (in minutes) to get empty after pumping starts?

   (1) A, 16.66 (2) C, 20 (3) D, 20 (4) D, 25

9. *[CAT 2004]* Two boats, traveling at 5 and 10 kms per hour, head directly towards each other. They begin at a distance of 20 kms from each other. How far apart are they (in kms) one minute before they collide?

   (1) 1/12 (2) 1/6 (3) 1/4 (4) 1/3

10. *[CAT 2004]* Karan and Arjun run a 100-metre race, where Karan beats Arjun by 10 metres. To do a favour to Arjun, Karan starts 10 metres behind the starting line in a second 100-metre race. They both run at their earlier speeds. Which of the following is true in connection with the second race?

    (1) Karan & Arjun reach finishing line together. (2) Arjun beats Karan by 1 metre.

    (3) Arjun beats Karan by 11 metres. (4) Karan beats Arjun by 1 metre.

11. *[CAT 2004]* If a man cycles at 10 km/hr, then he arrives at a certain place at 1 p.m. If he cycles at 15 km/hr, he will arrive at the same place at 11 a.m. At what speed must he cycle to get there at noon?

    (1) 11 km/hr (2) 12 km/hr (3) 13 km/hr (4) 14 km/hr

Note: The following question requires basic geometry awareness.

12. *[CAT 2004]* A sprinter starts running on a circular path of radius *r* metres. Her average speed (in metres/minute) is πr during the first 30 seconds, $\pi r / 2$ during next one minute, $\pi r / 4$ during next 2 minutes, $\pi r / 8$ during next 4 minutes, and so on. What is the ratio of the time taken for the n^{th} round to that for the previous round?

    (1) 4 (2) 8 (3) 16 (4) 32

Note: The following question requires very high geometry awareness.

13. *[CAT 2003 – Retest]* Two straight roads R_1 and R_2 diverge from a point A at an angle of 120°. Ram starts walking from point A along R_1 at a uniform speed of 3 km/hr. Shyam starts walking at the same time from A along R_2 at a uniform speed of 2 km/h. They continue walking for 4 hours along their respective roads and reach points B and C on R_1 and R_2, respectively. There is a straight line path connecting B and C.

    Then Ram returns to point A after walking along the line segments BC and CA. Shyam also returns to A after walking along line segments CB and BA. Their speeds remain unchanged. The time interval (in hours) between Ram's and Shyam's return to the point A is:

    (1) $\frac{10\sqrt{19}+26}{3}$ (2) $\frac{2\sqrt{19}+10}{3}$ (3) $\frac{\sqrt{19}+26}{3}$ (4) $\frac{\sqrt{19}+10}{3}$

14. *[CAT 2003 – Leaked]* The question is followed by two statements, A and B. Mark your answer as

    (1) if the question can be answered by using one of the statements alone but not by using the other statement alone
    (2) if the questions can be answer by using either of the statements alone.
    (3) if the question can be answered by using both statements together but not by either statements alone.
    (4) if the question cannot be answered on the basis of the two statements.

    If A and B run a race, then A wins by 60 seconds. If B and C run the same race, then B wins by 30 seconds. Assuming that C maintains a uniform speed, what is the time taken by C to finish the race?

    A: A and C run the same race and A wins by 375 meters
    B: The length of the race is 1 km

15. *[CAT 2003 – Leaked]* In a 4000 meter race around a circular stadium having a circumference of 1000 meters, the fastest runner and the slowest runner reach the same point at the end of the 5th minute, for the first time after the start of the race. All the runners have the same starting point and each runner maintains a uniform speed throughout the race. If the fastest runner runs at twice the speed of the slowest runner, what is the time taken by the fastest runner to finish the race?

    (1) 20 min (2) 15 min (3) 10 min (4) 5 min

16. *[CAT 2002]* Only a single rail track exists between station A and B on a railway line. One hour after the north bound super fast train N leaves station A for Station B, a south bound passenger train S reaches station A from station B. The speed of the super fast train is twice that of a normal express train E, while the speed of a passenger train S is half that of E. On a particular day N leaves for station B from Station A, 20 minutes behind the normal schedule. In order to maintain the schedule both N and S increased their speed. If the super fast train doubles its speed, what should be the ratio (approximately) of the speed of passenger train to that of the super fast train so that passenger train S reaches exactly at the scheduled time at station A on that day.

    (1) 1 : 3 (2) 1 : 4 (3) 1 : 5 (4) 1 : 6

17. *[CAT 2002]* A train approaches a tunnel AB. Inside the tunnel is a cat located at a point that is 3/8 of the distance AB measured from the entrance A. When the train whistles the cat runs. If the cat moves to the entrance of the tunnel, A, the train catches the cat exactly at the entrance. If the cat moves to the exit, B, the train catches the cat at exactly the exit. The speed of the train is greater than the speed of the cat by what order?

    (1) 3 : 1 (2) 4 : 1 (3) 5 :1 (4) None of these

18. *[CAT 2002]* 3 small pumps and a large pump are filling a tank. Each of the three small pumps works at 2/3rd the rate of the large pump. If all 4 pumps work at the same time, they should fill the tank in what fraction of the time that it would have taken the large pump alone?

    (1) 4/7 (2) 1/3 (3) 2/3 (4) 3/4

19. *[CAT 2002]* It takes 6 technicians a total of 10 hours to build a new server from Direct Computer, with each working at the same rate. If six technicians start to build the server at 11.00 AM, and one technician per hour is added beginning at 5.00 PM, at what time will the server be complete?

    (1) 6:40 pm (2) 7:00 pm (3) 7:20 pm (4) 8 pm

20. *[CAT 2002]* On a 20 km tunnel connecting two cities A and B there are three gutters. The distance between gutter 1 and 2 is half the distance between gutter 2 and 3. The distance from city A to its nearest gutter, gutter 1 is equal to the distance of city B from gutter 3. On a particular day the hospital in city A receives information that an accident has happened at the third gutter. The victim can be saved only if an operation is started within 40 minutes. An ambulance started from city A at 30 km/hr and crossed the first gutter after 5 minutes. If the driver had doubled the speed after that, what is the maximum amount of time the doctor would get to attend the patient at the hospital?

    Assume 1 minute is elapsed for taking the patient into and out of the ambulance.

    (1) 4 minutes (2)2.5 minutes (3) 1.5 minutes (4) Patient died before reaching the hospital

21. *[CAT 2001]* A can complete a piece of work in 4 days. B takes double the time taken by A, C takes double that of B, and D takes double that of C to complete the same task. They are paired in groups of two each. One pair takes two-thirds the time needed by the second pair to complete the work. Which is the first pair?

    (1) A, B (2) A, C (3) B, C (4) A, D

22. *[CAT 2001]* At his usual rowing rate, Rahul can travel 12 miles downstream in a certain river in six hours less than it takes him to travel the same distance upstream. But if he could double his usual rowing rate for this 24 mile round trip, the downstream 12 miles would then take only one hour less than the upstream 12 miles. What is the speed of the current in miles per hour?

    (1) 7/3 (2) 4/3 (3) 5/3 (4) 8/3

23. *[CAT 2001]* Shyama and Vyom walk up an escalator (moving stairway). The escalator moves at a constant speed. Shyama takes three steps for every two of Vyom's steps. Shyama gets to the top of the escalator after having taken 25 steps, while Vyom (because his slower pace lets the escalator do a little more of the work) takes only 20 steps to reach the top. If the escalator were turned off, how many steps would they have to take to walk up?

    (1) 40 (2) 50 (3) 60 (4) 80

24. *[CAT 2001]* There's a lot of work in preparing a birthday dinner. Even after the turkey is in the oven, there's still the potatoes and gravy, yams, salad, and cranberries, not to mention setting the table.

    Three friends, Asit, Arnold, and Afzal, work together to get all of these chores done. The time it takes them to do the work together is six hours less than Asit would have taken working alone, one hour less than Arnold would have taken alone, and half the time Afzal would have taken working alone.

    How long did it take them to do these chores working together?

    (1) 20 minutes (2) 30 minutes (3) 40 minutes (4) 50 minutes

25. *[CAT 2001]* A train X departs from station A at 11.00 a.m. for station B, which is 180 km away. Another train Y departs from station B at 11.00 a.m. for station A. Train X travels at an average speed of 70 kms/hr and does not stop anywhere until it arrives at station B. Train Y travels at an average speed of 50 kms/hr, but has to stop for 15 minutes at station C, which is 60 kms away from station B enroute to station A. Ignoring the lengths of the trains, what is the distance, to the nearest km, from station A to the point where the trains cross each other?

    (1) 112 (2) 118 (3) 120 (4) None of these

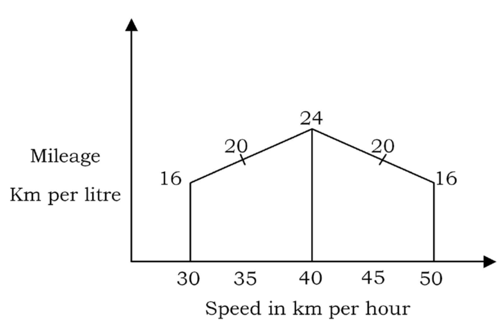
26. *[CAT 2001]* Three runners A, B and C run a race, with runner A finishing 12 meters ahead of runner B and 18 meters ahead of runner C, while runner B finishes 8 meters ahead of runner C. Each runner travels the entire distance at a constant speed. What was the length of the race?

    (1) 36 meters (2) 48 meters (3) 60 meters (4) 72 meters

27. *[CAT 1999]* Navjivan Express from Ahmedabad to Chennai leaves Ahmedabad at 6 : 30 am and travels at 50 km per hour towards Baroda situated 100 km away. At 7 : 00 am Howrah-Ahmedabad Express leaves Baroda towards Ahmedabad and travels at 40 km per hour. At 7 : 30 Mr. Shah, the traffic controller at Baroda realises that both the trains are running on the same track. How much time does he have to avert a head-on collision between the two trains?

    (1) 15 minutes (2) 20 minutes (3) 25 minutes (4) 30 minutes

Directions for questions 28 & 29: *[CAT 1999]* Rajiv reaches city B from city A in 4 hours, driving at the speed of 35 km per hour for the first 2 hours and at 45 km per hour for the next two hours. Aditi follows the same route, but drives at three different speeds 30, 40 and 50 km per hour, covering an equal distance in each speed segment. The two cars are similar with petrol consumption characteristics (km per litre) shown in the figure below.



28. The amount of petrol consumed by Aditi for the journey is

(1) 8.3 litres (2) 8.6 litres (3) 8.9 litres (4) 9.2 litres

29. Zoheb would like to drive Aditi's car over the same route from A to B and minimize the petrol consumption for the trip. The amount of petrol required by him is

(1) 6.67 litres (2) 7 litres (3) 6.33 litres (4) 6.0 litres

Answer Key

Ratio Proportion Variation - Exercise

1. 2	2. 2	3. 5	4. 2	5. 5	6. 1	7. 4	8. 1
9. 3	10. 2	11. 2	12. 4	13. 4	14. 4	15. 5	16. 4
17. 5	18. 2	19. 1	20. 3	21. 5	22. 5	23. 2	24. 2
25. 3	26. 1	27. 5	28. 3	29. 5	30. 4	31. 5	32. 3
33. 1	34. 4	35. 4	36. 1	37. 5	38. 4	39. 3	40. 3
41. 4	42. 2						

Ratio Proportion Variation - CAT Questions

1. 1	2. 4	3. 2	4. 3	5. 4	6. 2	7. 3	8. 2
9. 1							

Percentages – Exercise

1. 1	2. 5	3. 1	4. 4	5. 2	6. 3	7. 4	8. 2
9. 1	10. 2	11. 1	12. 5	13. 3	14. 5	15. 1	16. 4
17. 2	18. 3	19. 2	20. 5	21. 4	22. 1		

Profit Loss Discount – Exercise

1. 5	2. 2	3. 4	4. 4	5. 2	6. 5	7. 3	8. 3
9. 4	10. 1	11. 3	12. 5	13. 1	14. 4	15. 3	16. 4
17. 5	18. 4	19. 1	20. 3	21. 3	22. 4	23. 5	24. 4
25. 5	26. 1	27. 1	28. 5	29. 5	30. 3		

SI CI – Exercise

1. 2	2. 2	3. 4	4. 3	5. 3	6. 5	7. 5	8. 5
9. 3	10. 4	11. 4	12. 2	13. 1	14. 4	15. 1	16. 4
17. 3	18. 2	19. 1	20. 3	21. 2	22. 3	23. 5	24. 1

Percentages & Application – CAT Questions

1. 5	2. 1	3. 4	4. 3	5. 1	6. 3	7. 1	8. 2
9. 4							

Averages & Weighted Averages – Exercise

1. 4	2. 5	3. 2	4. 2	5. 2	6. 5	7. 1	8. 2
9. 5	10. 3	11. 4	12. 5	13. 1	14. 4	15. 2	16. 2
17. 4	18. 4	19. 1	20. 5	21. 5	22. 4	23. 1	24. 3
25. 5	26. 2	27. 3	28. 3				

Averages & Weighted Averages – CAT Questions

1. 1	2. 4	3. 3	4. 1	5. 3	6. 2	7. 1	8. 2
9. 3	10. 1						

Time Speed Distance – Exercise

1. 3	2. 3	3. 2	4. 5	5. 4	6. 2	7. 1	8. 3
9. 3	10. 5	11. 3	12. 5	13. 4	14. 1	15. 4	16. 1
17. 2	18. 1	19. 4	20. 5	21. 1	22. 3	23. 3	24. 1
25. 5	26. 3	27. 4	28. 2	29. 4	30. 3	31. 4	32. 2
33. 1	34. 2	35. 3	36. 4	37. 1	38. 4	39. 2	40. 2
41. 4	42. 2	43. 5	44. 4	45. 3	46. 1	47. 4	48. 3
49. 1	50. 1	51. 5	52. 1	53. 2			

Time and Work – Exercise

1. 1	2. 3	3. 2	4. 5	5. 3	6. 2	7. 5	8. 3
9. 3	10. 5	11. 5	12. 2	13. 4	14. 2	15. 2	

Time Speed Distance & Work – CAT Questions

1. 2	2. 3	3. 1	4. 3	5. 4	6. 2	7. 2	8. 3
9. 3	10. 4	11. 2	12. 3	13. 2	14. 1	15. 3	16. 4
17. 2	18. 2	19. 4	20. 3	21. 4	22. 4	23. 2	24. 3
25. 1	26. 2	27. 2	28. 3	29. 1			