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Answer Key





Terms of Algebra

Variable:

As discussed above, it is an alphabetic representation of a quantity that can assume different values.

Term, Coefficient and Degree:

Any variable (or its power) multiplied by a constant is a term.

The constant (along with its sign) with which the variable (or its power) is multiplied is said to be the coefficient

The sum of powers of all the variables present in the term is called the degree of the term. The relevance of degree is only in the case of polynomials, as will be learnt shortly. Till then it would just suffice to know that degree is defined only when the powers of variables are Whole numbers. Thus degree is not defined when the index of any of the variable is a fraction.

Examples:

Term	Term in	Coefficient	Degree
3 <i>x</i>	x	3	1
$-7y^{2}$	\mathcal{Y}^2	-7 (and not 7)	2
x^2y	x^2y	1	3
$\frac{abc}{2}$	abc	$\frac{1}{2}$	3
$\frac{3\sqrt{x}}{-4}$	\sqrt{x}	$-\frac{3}{4}$	Not defined
$\frac{2\sqrt{3}x^2}{y}$	$\frac{x^2}{y}$ or x^2y^{-1}	$2\sqrt{3}$	Not defined
-5	x^0 or a constant term	-5	0

Expressions and Polynomials:

Addition of two or more terms is an expression

Examples:
$$3x - 4$$
, $x^2 - y^2$, $x^3 - 3x^2y + 3xy^2 - y^3$, $x - \frac{1}{x}$, $x^2 - 5x + 6$, $x - 5\sqrt{x} + 6$, $2x + 1 - \frac{3}{x}$



Expressions wherein the powers of all variables are whole numbers are called Polynomials. Polynomials could have one or more than one variable but the index of each variable in each of the term has to be a whole number.

Thus from the above list of expressions 3x - 4 and $x^2 - 5x + 6$ are polynomials in one variable;

$$x^2 - y^2$$
, $x^3 - 3x^2y + 3xy^2 - y^3$ are polynomials in two variables; whereas $x - \frac{1}{x}$, $x - 5\sqrt{x} + 6$ and

$$2x+1-\frac{3}{x}$$
 are not polynomials.

Polynomials are classified on two basis: depending on the number of terms and depending on the degree. The degree of a polynomial is taken as the degree of the term having the highest degree.

Nomenclature depending on number of terms:

Number of terms Name Examples

1 Monomial
$$2x, -3x^2, 5, 2xy, -8abc, \frac{x}{3}$$

- 2 Binomial 3x 4, $x^2 y^2$, $x^2 4$, $\frac{xy}{5} \frac{1}{3}$
- 3 Trinomial $x^2 5x + 6$, a + b + c, $a^2 + 2ab + b^2$

Nomenclature depending on the degree of terms:

Degree Name Examples

- 0 Constant 3, $-\frac{1}{4}$
- 1 Linear $2x, \frac{x}{3}, 3x-5, 2-5y$
- 2 Quadratic $x^2 5x + 6, x^2 y^2, x^2 4, x + y + xy$
- 3 Cubic $-2x^3 + 3x^2 x + 5$, a + b + c + 3abc, $x^3 y^3$

Polynomials with degree 4 and 5 are called Quartic and Quintic respectively, but are not very popular in entrance exams. Further, in the following text, we will limit the scope of polynomial to imply a polynomial in one variable only as these are the most commonly occurring polynomial.

Any polynomial in one variable, say x, can also be thought of as a function of x and thus we will represent polynomials in general as f(x).



Roots of a polynomial:

Roots of a polynomial are those values of the variable for which f(x) = 0. Thus when we say root of 3x - 4 we imply the value of x that will make the value of 3x - 4 equal to 0 i.e. 3x - 4 = 0 for what value of x? Solving the equation, we get $x = \frac{4}{3}$ and this is the root of the polynomial

3x - 4.

Similarly roots of $f(x) = x^2 - 5x + 6$ are those values of x where f(x) = 0 i.e. $x^2 - 5x + 6 = 0$. Solving this quadratic equation we get the roots as x = 2 or 3. As a check, substitute x by 2 or 3 in the expression $x^2 - 5x + 6$ and see if the value indeed turns out to be zero.

Degree and Number of Roots of Polynomials

The degree of a polynomial is the number of roots that the polynomial can have.

Thus a linear polynomial will have just 1 root, while a quadratic polynomial will have two roots and a cubic polynomial will have three roots.

These are the total number of roots, they may be real or imaginary and they could also be distinct or identical.

Roots and the Graph of the Polynomial:

We have already seen that roots of a polynomial are those value of the variable for which the polynomial becomes equal to zero. Now, on the graph of the polynomial function, this point will be where the graph cuts the *X*-axis. At this point, the value of the polynomial or of f(x) is equal to zero as the values of the polynomial or f(x) are read off the *Y*-axis. Thus the values of the roots of a polynomial function are the *x*-coordinates of the points where the graph cuts the *X*-axis. The above will be clearer when we look at the graphs of certain common polynomials.

Graphs of most common polynomial functions:

The most commonly occurring polynomial functions are linear, quadratic and cubic. So it would be a good idea to get an idea of how these functions behave as *x* assumes values from $-\infty$ to $+\infty$.

Linear Polynomial:

The graph of a linear polynomial function will be a straight line. In fact this is the reason they are called 'linear'.

There would be two cases viz. a increasing line or a decreasing line (as x increases i.e. from left to right). This would depend whether the coefficient of x is positive or negative.



If the coefficient of x is positive, say 3x, as x increases, the value of 3x also increases and thus we will get a increasing line.

If the coefficient of x is negative, say -3x, when x is a small quantity (a negative value of large magnitude), -3x will be a large value (positive value of large magnitude). As x increases, the value of -3x will continuously decrease and we will get a decreasing line.

The above two cases would also be understood from the following graphs:



Slope of the Line:

Slope of the line is the inclination of the line and is defined as the change in the value of y for a change of 1 unit in x. If the change in y is large, its slope will have a larger magnitude and the line will be inclined steeper and if the change in the value of y for a unit change in x is very small, the slope will be small in magnitude and the line will be flatter. Further in an increase in x causes an increase in y, the slope is said to be positive and if an increase in x causes a decrease in y, the slope is said to be negative.

If one observes the working of finding the value of *y* for a corresponding value of *x*, one would realize that in the case of the above polynomial, for each unit change in *x*, *y* changes by 3 and thus this will be the slope of the line. Thus in the first case the slope = 3 and in the second case the slope is -3 (in increase in *x* is accompanied by a decrease in *y* and hence slope is negative).

Y-intercept:

Y-intercept is the point where the graph cuts the *Y*-axis. It is expressed as the *y*-coordinate of this point. At this point the value of *x* is zero.



The *y* intercept for a linear polynomial is the value of the constant term (as x = 0 at the *Y*-intercept) of the given polynomial.

Thus for any line y = ax + b, the slope of the line will be the value of *a* (along with the sign) and the *Y*-intercept will by *b* (along with the sign).

Quadratic Polynomial

The graph of a quadratic polynomial function will be a parabola, i.e. a \bigcup shaped graph. Just as we had a increasing line and a decreasing line we have two case here, that of a upright \bigcup and that of an inverted \bigcap .

Just as the sign of the coefficient of x in a linear equation decided if it would be a increasing line (coefficient being positive) or a decreasing line (coefficient being negative), similarly the coefficient of x^2 would decide if the graph would be an upright \bigcup (coefficient being positive) or an inverted \bigcap (coefficient being negative).



The following should be obvious after seeing the above graph and noticing the conditions for the graph to be an upright \bigcup or an inverted \bigcap :

For the polynomial function $y = ax^2 + bx + c$,

the graph will be \bigcup shaped when *a* is positive and in this case the polynomial function would have a specific minimum value that it can assume. The maximum value the polynomial can take will be ∞ and it will assume this value when the value of *x* is either very large or very small.



the graph will be \cap shaped when *a* is negative and in this case the polynomial function would have a specific maximum value that it can assume. The minimum value the polynomial will take is $-\infty$ and it will assume this value when the value of *x* is either very large or very small. Further, the roots are the points where the graph cuts the *X*-axis. When we study quadratic

equations we would see that the graph need not always cut the X-axis. When we study quadratic equations we would see that the graph need not always cut the X-axis. The placement of the graph on the X-Y plane would be studied in depth in the topic on quadratic equations.

Cubic Polynomial:

Again two shapes are possible for the graph depending on whether the coefficient of x^3 is positive or negative. The following graph shows the general graph of a cubic polynomial function:



The difference when *a* is positive and when it is negative should be understood clearly and the above graph should not be just mugged up.

If *a* is positive, as *x* assumes large positive values, the value of ax^3 will be a much larger positive value and consequently the polynomial function will also assume a large positive value. Thus as *x* assumes larger and larger positive values i.e. on the right most part of the graph, the graph will increase, move upwards.

(If you are wondering then why isn't the entire graph an upward sloping curve and why are there troughs and crests, they are there because of the presence of the other terms bx^2 , cx and d. When any of b, c and d are negative they would tend to decrease the value of the polynomial, for x being positive. However for *large* positive values of x, i.e. in the right hand part of the graph, the value of ax^3 will be far far more than the values of bx^2 and cx. So even if bx^2 , cx and d are negative, the larger value of ax^3 will more than compensate for the negative values of bx^2 ,



cx and *d*. But when *x* is small, this may not be the case and hence troughs and crests are formed when *x* has lower values.)

If *a* is negative, as *x* assumes large positive values, the value of ax^3 will be a negative value of a large magnitude and consequently the polynomial function will also assume a negative value of large magnitude. Thus as *x* assumes larger and larger positive values i.e. on the right most part of the graph, the graph will decrease, move downwards.

One should also note that the above shown graphs are just general graphs. While overall the shape would be similar, the placement of the graph on the X-Y plane may be different. And even the number of real roots need not always be 3.3 is the maximum roots possible, but as shown below, but could also be less.

The troughs and crests could also be very smoothed out affairs and not necessarily as shown in the earlier graph. (see the right hand graph below)





Algebraic Identities

Identity 1: $(a + b)^2 = a^2 + 2ab + b^2$, $(a - b)^2 = a^2 - 2ab + b^2$ These can be arranged to link $(a + b)^2$ and $(a - b)^2$ as follows: $(a + b)^2 = (a - b)^2 + 4ab$ Identity 2: $(a+b) \times (a-b) = a^2 - b^2$ Identity 3: $(x + a) \times (x + b) = x^{2} + x(a + b) + ab$ Identity 4: $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$ Identity 5: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ Identity 6: $a^{3} + b^{3} = (a + b) \times (a^{2} - ab + b^{2})$ $a^{3}-b^{3}=(a-b)\times(a^{2}+ab+b^{2})$ Re-arranging Identity 5, the above can also be written as: $a^{3} + b^{3} = (a + b)^{3} - 3ab(a + b)$ $a^{3}-b^{3}=(a-b)^{3}+3ab(a-b)$

Identity 7:

 $a^{3} + b^{3} + c^{3} = (a + b + c) \times (a^{2} + b^{2} + c^{2} - ab - bc - ac) + 3abc$ If (a + b + c) = 0, then $a^{3} + b^{3} + c^{3} = 3abc$



Exercise

1.	If $x^8 + \frac{1}{x^8} = 2207$, find the value of $x + \frac{1}{x}$.				
	(1) 3	(2) 5	(3) 7	(4) 9	
2.	If $x + \frac{1}{x} = 2$, find the v	alue of $x^9 + \frac{1}{x^9}$.			
	(1) –2	(2) 2	(3) –1	(4) 1	
3.	If $a + b = 4$ and $ab = 3$.75, find the value of $ a $	-b .		
	(1) 0.5	(2) 1	(3) 1.5	(4) 2	
4.	If $a^2 + b^2 = 0.5$ and $a + b^2 = 0.5$	b = 0.8, find the value of	of $ a-b $.		
	(1) 0.1	(2) 0.2	(3) 0.5	(4) 0.6	
5.	If $a + b = ab = 5$, find the value of $a^3 + b^3$.				
	(1) 125	(2) 25	(3) 50	(4) 250	
6.	$a + b + c = 5$ and $ab + bc + ac = 12$, find the value of $a^2 + b^2 + c^2$				
	(1) 0	(2) 1	(3) 2	(4) 4	
7.	If $a^2 + b^2 - 2a - 2b + 2ab = 15$, find the value of $a + b$.				
	(1) 1	(2) 3	(3) –5	(4) -3 or 5	
Dir	ections for questions 8	and 9:			
	a+b+c=3	$a^2 + b^2 + c^2 = 5$	abc = 8		
8.	Find the value of $\frac{1}{a} + \frac{1}{a}$	$\frac{1}{b} + \frac{1}{c}$			
	(1) 1	(2) 2	(3) 1/4	(4) 1/3	
9.	Find the value of $a^3 + b^3 + c^3$.				
	(1) 7	(2) 9	(3) –9	(4) 33	
10.	$11^3 + 22^3 - 33^3$ is divisible by which of the following				
	(1) 13	(2) 27	(3) 37	(4) 99	





We have already seen in Arithmetic, the process of writing a number as a product of its prime factors is called Factorisation. Similarly, in Algebra, to factorise a given expression is to write the given expression as a product of its factors.

Thus the factorised form of $x^2 - 5x + 6$ is $(x - 2) \times (x - 3)$. You can check by expanding the product that the two forms are identical. So the immediate question that arises is how does one identify the factors of a given expression? Factor theorem comes to our rescue.

Factor Theorem: Identifying a factor

Factor theorem states that (x - a) is a factor of any polynomial f(x) if the value of the polynomial is zero when x = a. If the value is not zero, it is not a factor.

Thus, say we want to find the factors of say $x^3 + 4x^2 - 11x - 30$.

Substituting x = 1, the value of the polynomial is $1 + 4 - 11 - 30 \neq 0$. Thus (x - 1) is not a factor.

Trying x = 2, the value is $8 + 4 \times 4 - 11 \times 2 - 30 = 8 + 16 - 22 - 30 \neq 0$. Thus (x - 2) is also not a factor.

Trying x = 3, the value is $27 + 4 \times 9 - 11 \times 3 - 30 = 27 + 36 - 33 - 30 = 0$. Thus (x - 3) is a factor of the expression.

This is the basic working of a factor theorem.

If we want to see if (x + 1) is a factor, you would have to substitute the value of x as -1 because (x + 1) = (x - (-1)) and comparing this with (x - a) we get a = -1.

Thus using the Factor theorem, you can identify if (x - a) is a factor of the given expression or not. But factorizing the given expression using this method would be very time-consuming as you will have to keep trying different values till you get all the factors. There is a way out for factorizing as we will learn shortly, but right now just focus on the factor theorem

Why does the factor theorem work?

Lets say that (x - a) is a factor of f(x). Then f(x) can be written as a product of (x - a) and some other expression, say g(x). Thus, we have

 $f(x) = (x - a) \times g(x).$

Now this relation is true for all values of *x* i.e. when *x* assumes any value, the equality will still remain. So if *x* assumes the value *a*, the right hand side will be zero and since it is equal to the left hand side, the value of f(x) when x = a will also be zero if (x - a) is a factor of f(x).



E.g. 1: If (x + 2) is a factor of $2x^3 - 3x^2 + k$, find the value of k. Since (x + 2) is a factor, hence when x = -2, the polynomial will be equal to zero. Thus, $2 \times (-2)^3 - 3 \times (-2)^2 + k = 0$ $-16 - 12 + k = 0 \implies k = 28$

Remainder theorem:

The remainder when f(x) is divided by (x - a) is the value of the polynomial when x assumes the value a.

Thus, this is a broader version of the Factor theorem itself. If on x assuming the value a, the polynomial becomes zero, the remainder theorem states that f(x) on being divided by (x - a), the remainder will be zero which is same as (x - a) being a factor of f(x).

But the remainder theorem has a utility of its own because if (x - a) is not a factor, we also get to know the value of the remainder.

E.g. 2: Find the remainder when $x^3 - 2x^2 + 3x - 4$ is divided by (x + 1)?

Putting x = -1, the value of the polynomial becomes $(-1)^3 - 2 \times (-1)^2 + 3(-1) - 4 = -1 - 2 - 3 - 4 = -10$. Thus when $x^3 - 2x^2 + 3x - 4$ is divided by (x + 1) the remainder is -10.

Factorising a given polynomial

The degree of a polynomial has a relevance here too. The degree of a polynomial is equal to the number of factors the polynomial can be factorised to. Thus a quadratic can be written as a product of two factors whereas a cubic polynomial can be written as a product of three factors.

Factorising a Quadratic:

Consider factorizing $x^2 - x - 6$

A quadratic is factorised by writing the term in x as a sum (or difference) of two terms in x. The process is first explained here and then we will learn how to identify which two terms should the term in x be broken into.

$$x^{2} - x - 6 = x^{2} - 3x + 2x - 6$$
$$= x(x - 3) + 2(x - 3)$$
$$= (x - 3) \times (x + 2)$$

The entire above working HAS to be done orally if you want to have any competitive edge in CAT. So make sure you learn the following working carefully...

We know that any quadratic expression can be factorised into two terms i.e.



 $x^{2} - x - 6 = (x + p) \times (x + q)$

All we need to know are the values of p and q. Expanding the RHS, we have

 $x^{2} - x - 6 = x^{2} + (p + q)x + pq$

Thus the two numbers that we are searching for, p and q, are such that their sum is -1 and product is -6.

Always start with finding the two numbers that satisfy the product. Two numbers that multiply to -6 are (-3, 2) or (3, -2) or (-6, 1) or (6, -1). The sum of which of these pairs is -1? Obviously the required values of p and q are -3 and 2. Thus, we can directly write the factorised form as

 $x^2 - 5x - 6 = (x - 3) \times (x + 2)$

E.g. 3: Factorise $x^2 - 10x - 24$

We need to find p and q such that p + q = -10 and pq = -24.

24 can be written as a product of 6×4 and 12×2 and 8×3 . Knowing that one of them has to be positive and one has to be negative (because the product is negative) and also that the sum has to be -10, the values can be identified as -12 and 2. Thus,

 $x^2 - 10x - 24 = (x - 12) \times (x + 2)$

The above is very easy and can be mastered very easily. The problem comes when the coefficient of x^2 is any value other than 1. In such cases, use the following approach...

E.g. 4: Factorise $4x^2 + 9x + 5 = 0$

Here we should NOT look out for two numbers whose product is 5 and sum is 9. Instead we should look out for two numbers such that whose product is 5×4 i.e. 20 (the constant term multiplied with coefficient of x^2) and the sum is 9. The numbers are 5 and 4. Now the factors are NOT going to be (x + 5) and (x + 4) but the just

found values have to be divided by the coefficient of x^2 i.e. the factors are $\left(x + \frac{5}{4}\right)$

and $\left(x + \frac{4}{4}\right)$. Also do not forget that the factorised form will be the coefficient of x^2

multiplied with the just found factors. Thus,

$$4x^2 + 9x + 5 = 4 \times \left(x + \frac{5}{4}\right) \times \left(x + 1\right)$$
 and the roots are $-\frac{5}{4}$ and -1 .



Factorising a Cubic

While there exists a process like the above to factorise a cubic expression also, it is very cumbersome to find the numbers orally as we have to satisfy many conditions simultaneously. Thus to factorise a cubic expression, we take help of the factor theorem and first identify one factor by hit and trial. After finding one factor, the given expression can be reduced to a product involving the just found factor and a quadratic. The quadratic can then be factorised as learnt above.

E.g. 5: Factorise $x^3 + 4x^2 - 11x - 30$

Obviously x = 1, would not make the polynomial equal to zero.

Trying, x = -1, the polynomial is -1 + 4 + 11 - 30 which is gain not equal to zero.

x = 2 will also not make the polynomial equal to zero as the negative quantities are far larger than the positive quantities.

Try x = -2, the polynomial becomes -8 + 16 + 22 - 30 = 0. Thus (x - (-2)) i.e. (x + 2) is a factor.

Now the given cubic expression can be written as,

 $x^3 + 4x^2 - 11x - 30 = (x+2) \times g(x)$

It should be obvious that g(x) should be a quadratic expression, only then would the RHS have terms in x^3 , x^2 , x and a constant term and the two could be equal.

Thus, $x^3 + 4x^2 - 11x - 30 = (x + 2) \times (ax^2 + bx + c)$ and we have to find the values of *a*, *b*, and *c*.

One way is to actually divided $x^3 + 4x^2 - 11x - 30$ by (x + 2) and find the quadratic. But then this is time consuming. One can find the values of *a*, *b*, and *c* by just equating the coefficients of the like terms of LHS and RHS.

On the LHS we have x^3 and thus in the product of the two terms on the RHS also we should have x^3 . The only way of getting a term in x^3 on the RHS is by multiplying the term of x from the factor (x + 2) with the term ax^2 in the factor $(ax^2 + bx + c)$.

Term in
$$x^3$$

$$(x+2) \times (ax^2 + bx + c)$$
$$(x \times ax^2)$$

Thus $x^3 = ax^3$ i.e. a = 1

Similarly, on the LHS the constant is -30 and thus in the product of the two factors on the RHS also, the constant should be -30. The only way of getting a constant term on the RHS is by multiplying the constant term of the factor (x + 2) with the constant term in the factor $(ax^2 + bx + c)$.



$$(x+2) \times (ax^2 + bx + c)$$
Constant Term
$$2 \times c$$

Thus 2c = -30 i.e. c = -15.

To find *b*, you can make use of the above found values of *a* and *c* and equate either of the term, x^2 or *x*, of the two sides of the equality.

On the LHS the term in x^2 is $4x^2$ and thus in the product of the two factors on the RHS also, the term in x^2 should be $4x^2$. In the product there are two ways of getting a term in x^2 , as shown below:

Term in
$$x^2$$

 $(x+2) \times (x^2 + bx - 15)$
 $x \times bx$

Thus $2x^2 + bx^2 = 4x^2$ i.e. b = 2. Thus, $x^3 + 4x^2 - 11x - 30 = (x + 2) \times (x^2 + 2x - 15)$ $= (x + 2) \times (x + 5) \times (x - 3)$

E.g. 6: Factorise $2x^3 - 3x^2 - 17x - 12$.

To start with we have to use factor theorem to identify the first factor. For x = -1, the polynomial will be -2 - 3 + 17 - 12 = 0. Thus (x + 1) is a factor of $2x^3 - 3x^2 - 17x - 12$.

Now, $2x^3 - 3x^2 - 17x - 12 = (x + 1) \times (ax^2 + bx + c)$

Equating the terms in x^3 on the two sides; $2x^3 = ax^3$ i.e. a = 2

Equating the constant terms on the two sides; -12 = c i.e. c = -12

Thus, $2x^3 - 3x^2 - 17x - 12 = (x + 1) \times (2x^2 + bx - 12)$

Equating the term in x^2 on the two sides; $-3x^2 = bx^2 + 2x^2$ i.e. b = -5.

Thus, $2x^3 - 3x^2 - 17x - 12 = (x + 1) \times (2x^2 - 5x - 12)$

To factorise the quadratic, we need to find two numbers whose product is -24 and sum is -5. A little trial and error will give the values to be -8 and 3.

Thus,
$$2x^3 - 3x^2 - 17x - 12 = (x+1) \times 2 \times (x-4) \times \left(x - \frac{3}{2}\right)$$
 or $(x+1) \times (x-4) \times (2x-3)$



Exercise

1. Factorise each of the following:

i. $x^3 + 2x^2 - 11x - 12$

ii. $x^3 - 2x^2 - 9x + 18$

iii.
$$6x^3 + 5x^2 - 2x - 1$$

(No options for this question)

- 2. What is the value of k for which the expression $x^3 + kx^2 + 3x + 4$ is divisible by (x + 3)?
 - (1) 14/9 (2) 22/9 (3) 32/9 (4) 40/9
- 3. What is the remainder when $x^3 2x^2 + 3x 4$ is divided by (x + 1)?
 - (1) -10 (2) -6 (3) -2 (4) -1
- 4. Find the value of *a* if the remainder when $(x + a) \times (x^2 ax a^2)$ is divided by (x a) is 54.
 - (1) -1 (2) 0 (3) 2 (4) -3

5. What is the value of k - l if the expression $kx^3 - 4x^2 - lx + 5$ is divisible by $(x^2 - x - 2)$?

- (1) -1 (2) 1 (3) -9 (4) 9
- 6. If (x + 2) is a common factor of $x^3 + ax^2 7x 6$ and $x^3 + 4x^2 + bx 6$, find the value of a + b.
 - (1) -1 (2) 0 (3) 1 (4) 2

7. What is the common factor of the polynomials $6x^3 - 31x^2 + 4x + 5$ and $6x^3 - 29x^2 - 6x + 5$?

- (1) 2x + 1 (2) 2x 1 (3) x 5 (4) x + 5
- 8. If $x^4 + mx^3 + nx^2 + 5x 6 = (x^2 x + 2) \times (px^2 + qx + r)$, find the value of (p + q + r) (m + n).
 - (1) -1 (2) 0 (3) 1 (4) 2



9. i. (x + y) is a factor of $x^n + y^n$ for natural values of *n* that are

	(1) Odd	(2) Even	(3) All values of n	(4) No value of n		
	ii. $(x - y)$ is a factor of $x^n + y^n$ for natural values of <i>n</i> that are					
	(1) Odd	(2) Even	(3) All values of n	(4) No value of n		
	iii. $(x + y)$ is a factor of $x^n - y^n$ for natural values of <i>n</i> that are					
	(1) Odd	(2) Even	(3) All values of n	(4) No value of n		
	iv. $(x - y)$ is a factor of $x^n - y^n$ for natural values of <i>n</i> that are					
	(1) Odd	(2) Even	(3) All values of n	(4) No value of n		
10.	i. If $x^7 + y^7 = (x + y) \times f(x)$, then the polynomial $f(x)$ will be:					
	ii. If $x^7 - y^7 = (x - y) \times g(x)$, then the polynomial $g(x)$ will be:					

The answer options for both the above are:

- (1) $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$
- (2) $x^6 x^5y + x^4y^2 x^3y^3 + x^2y^4 xy^5 + y^6$
- $(3) x^6 + 7x^5y + 21x^4 y^2 + 35x^3 y^3 + 21x^2 y^4 + 7x y^5 + y^6$
- (4) $x^6 7x^5y + 21x^4y^2 35x^3y^3 + 21x^2y^4 7xy^5 + y^6$



Appendix: Multiplying factors efficiently

Now let's look at a method of doing the calculations orally. Taking an example:

$$(3x + 4) \times (2x - 5) = 3x \times 2x - 3x \times 5 + 4 \times 2x - 4 \times 5$$
$$= 6x^{2} - 15x + 8x - 20$$
$$= 6x^{2} - 7x - 20$$

One should be able to write this final product in one step directly and that too by doing all calculation work orally. For this, pay attention to the following carefully.

The product of $(x + a) \times (x + b)$ has a term in x^2 , one term in x and one constant term (devoid of x).

The term of x^2 is got by multiplying the two terms of x present one each in the two factors.

$$(3x+4) \times (2x-5)$$

$$6x^2$$

The term in x is got by two products as depicted in the following picture. The sum of the two products is your final term in x. While taking the product and sum, be very particular about the signs.

$$(3x+4) \times (2x-5)$$

$$(3x+4) \times ($$

The constant term is got by just multiplying the constant terms of both the brackets...

$$(3x+4) \times (2x-5)$$

Thus the product $(3x + 4) \times (2x - 5)$ can be directly written as $6x^2 - 7x - 20$.

A similar process as above can be used to find the product of more than two factors, say

 $(2x-1) \times (x+4) \times (4x+3)$

Let us first do it by the longer approach that quite a few of us are accustomed to and then learn the shorter approach. Multiplying the last two brackets using the shorter approach just learnt...



$$(2x-1) \times (x+4) \times (4x+3) = (2x-1) \times (4x^2 + 19x + 12)$$

= $2x \times 4x^2 + 2x \times 19x + 2x \times 12 - 4x^2 - 19x - 12$
= $8x^3 + 38x^2 + 24x - 4x^2 - 19x - 12 = 8x^3 + 34x^2 + 5x - 12$

Thus we see that in a product hat involves three factors, we have a term in x^3 , one term in x^2 , one in x and one constant term. The following four diagrammatic representation tells us how these four final terms are obtained.

In the above, multiplication, observe closely and we have each of the following multiplication of three terms

$$2x \times x \times 4x \qquad 2x \times x \times 3 \qquad 2x \times 4 \times 4x \qquad 2x \times 4 \times 3$$

-1 \times x \times 4x
$$-1 \times x \times 3 \qquad -1 \times 4 \times 4x \qquad -1 \times 4 \times 3$$

Thus, while multiplying three factors, remember that we have to multiply three terms, one term each from the three factors, in all possible manners.

Term of x^3 is obtained in just one way – multiplying the three terms in x appearing in each of the three factors.

Term in x^2 is obtained by summing up three individual products. Each product is a multiplication of two terms in x from two of the factors and the constant term from the other factor...

$$(2x-1) \times (x+4) \times (4x+3) \qquad (2x-1) \times (x+3) \times (x+3) \times (x+3) \qquad (2x-1) \times (x+3) \times (x+3) \times (x+3) \qquad (2x-1) \times (x+3) \qquad (2x-1) \times (x+3) \qquad (2x-1) \times (x+3) \qquad (2x$$

Similarly the term in x is also obtained by the summation of three products. This time each product is a multiplication of one term in x from any one bracket and two constant terms from the other two brackets...

$$(2x-1) \times (x+4) \times (4x+3)$$

$$(2x-1) \times (x+3)$$

$$(2x-1) \times (x+3$$



To find the constant term is very easy. Just multiply all the constants together...

$$(2x-1) \times (x+4) \times (4x+3)$$

Do not memorise the above placements i.e. arrow position. Just remember the following:

For term in x^2 : Multiply each of the three constant term with the terms in x present in the other two brackets and add the three products.

For term in *x*: Multiply each of the three terms in *x* with the two constant present in the other two brackets and add the three products.

Thus the product,

$$(2x-1) \times (x+4) \times (4x+3) = 8x^3 - 34x^2 + 5x - 12$$

In terms of an equation, the above would boil down to the following:

$$(x + a) \times (x + b) \times (x + c) = x^{3} + (a + b + c) \times x^{2} + (ab + bc + ca) \times x + abc$$

Please note that in the above, the coefficients of x in each of the factor is 1 and hence it looks very simple or else it is exactly the same as we did above.

Practice:

Find the product orally in one step. You should write your answer in the space on the right of the question and use NO other space for your workng.

$1.(x+1) \times (x+3)$	2. $(x - 1) \times (x - 7)$
3. $(x + 2) \times (x - 5)$	4. $(2x - 1) \times (2x + 5)$
5. $(4x - 3) \times (3x + 4)$	6. $(x - 1) \times (x + 3) \times (x - 2)$
7. $(2x-1) \times (x-3) \times (x+4)$	8. $(3x - 2) \times (2x - 3) \times (5x - 4)$
9. $(1-x) \times (2x+1) \times (3x-3)$	10. $(1-x) \times (2-x) \times (3-x)$

$10x^3 + 6x^2 - 11x + 6$	$6^{-}-8x_{3}+6x_{5}-3$	$8^{\circ} 30x_3 - 86x_5 + 85x - 54$	$7.2x^3 + x^2 - 25x + 12$	$9 + x_{2} - x_{3} - x_{2}$
$5.12x^2 + 7x - 12$	$\zeta - x_8 + z_7 x_5 + \delta x_7$	$3. x^2 - 3x - 10$	$2^{-1}x^{2} - 8x + 7$	$\xi + x_{z} + 4x + 3$

:srowers:



Linear Simultaneous Equations

Consider the equation 2x - 3y = 15. This is obviously a linear equation (since the degree is one)

in two variables (x and y). The above equation could be re-arranged as $y = \frac{2}{3}x - 5$

This is exactly similar to a linear polynomial function that we have seen, y = ax + b, and we know that it represents a line.

The re-arranged form should also suggest that for any value of *x* that we choose, we can find a corresponding value of *y*. This pair of value of *x* and corresponding *y*, denoted by $\{x, y\}$, would satisfy the equation 2x - 3y = 15. Thus there could be infinite solutions to the equation 2x - 3y = 15.

Even graphically, we come to the same conclusion as any point on the line denoted by

 $y = \frac{2}{3}x - 5$ would satisfy the relation $y = \frac{2}{3}x - 5$ and thus it would also satisfy 2x - 3y = 15.



Any point on the line will satisfy 2x - 3y = 15

The same is valid for any linear equation of the type ax + by = c. Thus for a linear equation in two variables, there would be infinite solutions. You could assume just any value for x (or y) and find the corresponding value for y (or x).

Consider another similar equation, 4x + y = 14. This equation would also have infinite solutions, by the same logic as used above. Also, every point on the line y = -4x + 14 would be a solution to the equation.

Now when the two equations are consider simultaneously, as seen in the figure below, while there are infinite points on each line and consequently infinite solutions to each equation independently, there is just one point common to both the lines, the point of intersection. Thus



the coordinates of this point would satisfy both the equations and this would be the only solution that satisfies both. No other solution of either equation would satisfy the other equation.



Thus, the simultaneous solution to the two equations is unique. When we talk of solving two linear equations in two variables simultaneously, we are talking about finding this unique solution that satisfies both the equations.

Finding the simultaneous solution:

Do two linear equation in two variable always have a unique simultaneous solution?

When faced with two linear simultaneous equation, do not immediately start the above learnt process of finding the unique solution. You may end up doing unnecessary work only to get stuck further ahead, as shown in the following example:

Inconsistent Equations, No unique Solution

E.g. 7: Solve for

```
6x - 10y = 15 .....(i) and
15x - 25y = -9 .....(ii)
```

Let's eliminate x. Making the coefficient of x equal to 30 in the two equations and then subtracting them, we get

(i)
$$\times 5$$
 30x - 50y = 75(iii)
(ii) $\times 2$ 30x - 50y = -18(iv)
(iii) - (iv) 0 = 93!?

Even if you try eliminating *y*, you would reach the same conclusion.

The problem is that not every two linear simultaneous equation has a unique simultaneous solution. It would save you precious time if you first ascertain whether a simultaneous solution exists or does not exist. Why some equations do have a simultaneous solution and other do not have is explained below and also the conditions to ascertain the case is given...



We already have understood that any linear equation in *x* and *y* represent a straight line on the X-Y plane.

When we try plotting the above two linear equations, we get the following figure...

$$6x - 10y = 15 \implies y = \frac{3}{5}x - \frac{3}{2}$$
 and $15x - 25y = -9 \implies y = \frac{3}{5}x + \frac{9}{25}$



Thus we see that the slope of the two lines is the same, $\frac{3}{5}$, and thus they are parallel to each

other. Thus there exists no common solution.

The reason we get the weird result (0 = 93) is because such equations (resulting in parallel lines) are called Inconsistent Equations. Just see the play on the coefficients of *x* and *y* in the two equations. The coefficient of *x* in equation (ii) is 2.5 times the coefficient of *x* in equation (i). And so is the coefficient of *y* in equation (ii), 2.5 times the coefficient of *y* in equation (i). Thus if we know that 6x - 10y = 15, multiplying the equation by 2.5, we get 15x - 25y = 37.5. If 15x - 25y is equal to 37.5, the second equation is Inconsistent with the first because it says 15x - 25y = -9. This is the inconsistency between the two equation.

Condition for two equation to be inconsistent:

Based on the above explanation, if we have two equations,

ax + by = c and px + qy = r,

and if $\frac{p}{a} = \frac{q}{b} = k$, then multiplying the second equation by k would make the coefficients of x

and y the same as that in the first equation. In this case if $\frac{r}{c} \neq k$ we will have an inconsistency.

Thus, the condition for Inconsistent Equations and no simultaneous solution is $\frac{p}{a} = \frac{q}{b} \neq \frac{r}{c}$.



Dependent Equations, Infinite simultaneous solution

What happens when $\frac{p}{a} = \frac{q}{b} = \frac{r}{c}$ in the case of above two equations?

In such a case, one equation is nothing but a multiple of the other equation. Thus, strictly speaking we do not have two equations but only one equation. And as such any solution of the first equation will also be a solution of the second equation. Since we already know, individually each equation has infinite solution, each of these would also be the solution for the other equation and the system of equation would have infinite simultaneous solutions. Read through the following example to understand the above...

E.g. 8: Solve for x and y

$$3x + 7y = 5$$
(i)
 $9x + 21y = 15$ (ii)
In this equation we see that $\frac{3}{9} = \frac{7}{21} = \frac{5}{15}$

Any solution to the equation (i), say x = 4 and y = -1, would also satisfy equation (ii) and it obviously has to since equation (ii) is same as equation (i) because we can divide both sides by 3 and the equation (ii) becomes equation (i).

Thinking contextually, if x were the cost of pens and y were the cost of pencils, equation (i) tells us that 3 pens and 7 pencils cost Rs. 5. So obviously thrice these quantities, 9 pens and 21 pencils will cost Rs. 15, so equation (ii) does not give us any new information, its just a restatement of equation (i).

Such equations are called Dependent equations and a system of dependent equations has Infinite solutions.

To capture all that we have learnt so far,





Linear Equations in three variables

One would already have a common idea that if the number of variables involved are n, we would need n equations to solve the system of equation. While the above is right, we need to be more precise about what we mean by solving the equations.

Consider the following equations:

Solve for x, y and z that satisfy the two equations simultaneously.

 $6x - 4y + z = 10 \dots(i)$

9x - 6y - z = 8(ii)

It would be wrong to say that we cannot solve the system of equations as it involves three variables and we have just two equations.

The correct answer would be that there would be infinite solutions for the systems of equations.

(i) + (ii) 15x - 10y = 18 (iii)

Taking any value of x, we can find a corresponding value of y that will satisfy the equation. And then plugging these values of x and y in either of the equation, we can get the value of z. E.g. If we assume x = 0, y = -1.8 from (iii).

Putting x = 0 and y = 1.8 in (i), we get $z = 10 + 4 \times (-1.8) = 2.8$

Substituting these values of x, y and z in eqn (ii), $9 \times 0 - 6 \times (-1.8) - 2.8 = 0 + 10.8 - 2.8 = 8$ i.e. they satisfy the two equations.

We could have started with any value of x and arrived at a solution of x, y and z that satisfy the two equations simultaneously. Thus there are infinite solutions to the system of equation.

Another point worth noting in the above equations is that while we cannot find any unique solution to the two equations, we can definitely find the solution to z. This is because if we look at the coefficients of x and y in the two equations carefully, we realize that they have the same

ratio i.e. $\frac{6}{9} = \frac{4}{6}$. Thus both x and y can be eliminated simultaneously...

(i)
$$\times 1.5 \quad 9x - 6y + 1.5z = 15 \dots$$
 (iv)

(iv) – (ii)
$$2.5z = 7 \implies z = \frac{7}{2.5} = 2.8$$

Thus, whatever may be the values of x and y, the value of z will always be 2.8

Similarly, in the above system of equation, we can also find the value of any multiple of (3x - 2y), by eliminating *z* or by substituting the value of *z* as 2.8 in either equation.

Thus do not be in a hurry to write-off any system of three equations with 2 variables are unsolvable. Entrance exams are notorious to have questions on the above aspect.



E.g. 9: At a certain fast food restaurant, Brian can buy 3 burgers, 7 shakes and one order of fries for Rs. 120 exactly. At the same place it would cost Rs. 164.5 for 4 burgers, 10 shakes and one order of fries. How much would it cost for an ordinary meal of one burger, one shake and one order of fries?

In the above questions, we can make the equations as given below:

3B + 7S + 1F = 120(i)

4B + 10S + 1F = 164.5(ii)

where B is the cost of one burger, S, the cost of one shake and F the cost of one order of fries.

We need to find the value of 1B + 1S + 1F.

(i) $3 \quad 9B + 21S + 3F = 360$(iii)

(ii) 28B + 20S + 2F = 329.....(iv)

(iii) - (iv) 1B + 1S + 1F = 31

Hence Rs. 31 is the answer.

Note here that you cannot find the individual cost of each item.

Exercise 3.1:

1. Find the value of x + y in each of the following case of simultaneous equations.

i.
$$12y = 5x - 1; \ 3x = 7y$$

(1) -4 (2) 4 (3) 10 (4) -10
ii. $\frac{4}{x} + \frac{3}{y} = 1; -\frac{2}{x} + \frac{1}{y} = 9$
(1) -247/30 (2) -5/6 (3) -6/5 (4) -30/247
iii. $9x^2 + 16y^2 = 40; \ x^2 - 4y^2 = 3$
(1) 2.5 or -2.5 (2) 1.5 or -1.5 (3) 2.5 (4) (1) or (2)
iv. $\frac{x+3}{5} = \frac{8-y}{4} = \frac{3(x+y)}{8}$
(1) 8 (2) 10 (3) 12 (4) 15



- 2. Find the value of *a* if the two given equations are consistent: 4x 3y = 0; $3x^2 + 4xy ay^2 = 0$
 - (1) 75 (2) 75/2 (3) 75/16 (4) 75/4
- 3. If the following system of simultaneous equations has no simultaneous solution, what is the value of *k*? Also what value of *n* is not acceptable?

3x - 2y = 6; kx + 9y = n

- (1) 27/2, 27 (2) -27/2, 27 (3) -27/2, -27 (4) 27/2, -27
- 4. For the simultaneous system of equations, x + 9y = 12; -4x + ky = m, to have infinite simultaneous solutions, what should be the value of k + m?
 - (1) 84 (2) -84 (3) 12 (4) -12
- 5. If 12 birds sit on each branch of a tree, 6 birds don't have a place to sit. If 15 birds sit on each branch, 2 branches are left vacant. The number of birds are how many more than the number of branches?
 - (1) 148 (2) 138 (3) 128 (4) 118
- 6. From a group of managers and assistants, 4 assistants leave. Now there is one assistant for every two managers. Next, 8 managers leave. Now there are three managers for every two assistants. The original number of managers are how many more than the original number of assistants?
 - (1) 8 (2) 10 (3) 12 (4) 15
- 7. The weight of A is the sum of the weights of B and C. If twice the weight of C is added to the weight of A and B, then you get 288. If thrice the weight of B is subtracted from the sum of the weights of A and C, then you get 6. What will you get if you subtract the weights of B and C from twice the weight of A?
 - (1) 114.6 (2) 58.8 (3) 55.8 (4) 96.4
- 8. If 2x + 3y = 4; 2z y = -6, 7x 9z = 11; find the value of x + y + z.
 - (1) 1 (2) 0 (3) -1 (4) Cannot be determined



Integer Solutions to One Linear Equation in Two Variables

Consider the following context that gives rise to a linear equation in two variables:

Each pencil costs Rs. 3 and each pen costs Rs. 4. I purchased few pencils and pens and spent a total of Rs. 40 on them. In how many different ways could I have made the purchase?

The data given boils down to the equation 3x + 4y = 40, where *x* is the number of pencils and *y* is the number of pens bought. We already know that this equation has infinite solutions. However, another implicit condition in this data is that the number of pencils and pens bought could only be whole numbers and not fractions. Thus *x* and *y* have to be whole numbers. Would the number of whole-number solutions to 3x + 4y = 40 also be infinite? Or would they be limited and if so how many whole-number solutions would exist?

One immediate whole-number solution to 3x + 4y = 4 is $\{0, 10\}$. One can identify this solution because 40 is a multiple of 4 and thus putting x = 0, we get y = 10. Are there other solutions and if yes, how can we find them with hit and trial? The following explanation is the answer...

It is obvious that if the number of pencils bought are increased, the number of pens bought will decrease and vice-versa. Thus if x increases, y should decrease and if x decreases, y should increase.

Not only can we establish the above relation between x and y, but we can also ascertain the quantum of increase (or decrease) in x that accompanies a quantum of decrease (or increase) in y.

For each pencil that is additionally bought (*x* increasing by 1), an additional amount of Rs. 3 is spent on pencils (3x increases by 3). Similarly if the number of pencils bought are increased by 2, 3, 4, 5, ..., the amount spent on pencils will be higher by Rs. 6, 9, 12, 15, ... respectively.

Further the increase in the amount spent on pencils should be equal to the decrease in the amount spent on pens as the total money spent (Rs. 40) is a constant.

When 1 less pen is purchased (y decreases by 1), the amount spent on pens decreases by Rs. 4 (4y decreases by 4). Similarly if the number of pens bought are decreased by 2, 3, 4, 5, ..., the amount spent on pens would be lower by Rs. 8, 12, 16, 20, ... respectively.

Since the decrease in the amount spent on pens has to be equal to the increase in the amount spent on pencils, the only value common to the two series we just found is Rs. 12. In this case the number of pens purchased decreases by 3 and the number of pencils purchased increases by 4. Thus whenever the amount spent on pens is decreased by 12, the number of pens bought will decrease by 3 and this amount will be instead spent on pencils and with Rs. 12 more being spent on pencils, we can purchase 4 more pencils. Let's see the above by plugging the values in the equations to see if they satisfy the equation...



# of pencils bought, x	Amt spent on pencils, $3x$	Amount spent on pens, $4y$	# of pens bought, y
0	0	40	10
1	3	37	Not a whole no.
2	6	34	Not a whole no.
3	9	31	Not a whole no.
4	12	28	7
5	15	25	Not a whole no.
6	18	22	Not a whole no.
7	21	19	Not a whole no.
8	24	16	4
9	27	13	Not a whole no.
10	30	10	Not a whole no.
11	33	7	Not a whole no.
12	36	4	1
13	39	1	Not a whole no.

Thus the whole number solutions to the equation 3x + 4y = 40 are $\{0, 10\}$, $\{4, 7\}$, $\{8, 4\}$ and $\{12, 1\}$

The learning for whole-number solutions to the equation 3x + 4y = 40 are captured below:

1. The number of pencils bought (x) differ in consecutive solutions by the coefficient of the other variable (y) i.e. 4. Thus the number of pencils bought (0, 4, 8, 12) is an AP with common difference being 4

2. The number of pen bought (y) differ in consecutive solutions by the coefficient of the other variable (x) i.e. 3. Thus the number of pen bought (10, 7, 4, 1) is an AP with common difference being 3 (or correctly -3, depends on how you see 10, 7, 4, 1 or 1, 4, 7, 10)

3. The amounts spent on either pen or pencil (3x or 4y) differ in consecutive solutions by the product 3×4 i.e. 12.

The above learnings can be generalized for any equation and can also apply to integer solutions and not just whole-number solutions. Please read the following generalizations very carefully because when you genralise, you have to look out for few adjustments that you have to make:



1. The variable changing in consecutive solutions by the coefficient of the other variable holds for integer solutions also. Thus if we were looking for integer solutions to 3x + 4y = 40, the list of earlier found solutions $\{0, 10\}, \{4, 7\}, \{8, 4\}$ and $\{12, 1\}$ could be extended at both ends to give us infinite integer solutions as follows...

After {12, 1}, if x increases by 4 and y decreases by 3, we get x = 16 and y = -2, which again satisfy the equation 3x + 4y = 40 (check for yourself). Thus {16, -2} is also a solution, though an integer solution and not a whole number solution. So are {20, -5}, {24, -8}, ... and so on.

Also on the other end {0, 10}, if x decreases by 4, y should increase by 3 and we get x = -4, y = 13 which again satisfy the equation 3x + 4y = 40 (check for yourself). Thus {-4, 13} is also an integer solution, so are {-8, 16}, {-12, 19}, {-16, 22}, ... and so on.

Thus the integer solutions to the equation 3x + 4y = 40 are

 $\dots, \{-8, 16\}, \{-4, 13\}, \{0, 10\}, \{4, 7\}, \{8, 4\}, \{12, 1\}, \{16, -2\}, \{20, -5\}, \dots$

2. Had the equation been 3x - 4y = 40, the variables would still change in successive integer solutions by coefficient of the other variable, but this time an increase in x is accompanied with an increase in y and correspondingly a decrease in x is accompanied by a decrease in y. This is because now it says that the *difference* between 3x and 4y is a constant 40. Thus if 3x increases, 4y also has to increase to maintain the difference.

Thus once we find one solution by hit and trial, in this case $\{0, -10\}$, we can expand this on either sides as follows:

Increasing *x* by 4 and *y* by 3, we get successive solutions as $\{4, -7\}$, $\{8, -4\}$, $\{12, -1\}$, $\{16, 2\}$, $\{20, 5\}$, Check for yourself that each of these solutions satisfy the equation.

Similarly decreasing x by 4 and y by 3, we get further solutions as $\{-4, -13\}, \{-8, -16\}, \dots$

Thus the integer solutions to 3x - 4y = 40 are

 $\ldots, \{-8, -16\}, \{-4, -13\}, \{0, -10\}, \{4, -7\}, \{8, -4\}, \{12, -1\}, \{16, 2\}, \{20, 5\}, \ldots$

Please not that in this case there would be infinite whole-number solutions as well unlike earlier case where there were limited whole number solutions because an increase in one variable caused a decrease in other variable.

3. Do not blindly apply the funda that values of the variables are going to change in successive integer solutions by the coefficient of the other variable.

E.g. For the equation 4x - 6y = 10, one integer solution is $\{4, 1\}$.

(Later we shall see how to find one solution by an efficient process rather than random hit and trial)

In this case, are we right in saying that in successive solutions x and y will both increase or both decrease by 6 and 4 respectively? Let's see...



Starting from {4, 1}, increasing both *x* and *y* by the above values we arrive at solutions {10, 5}, {16, 9}, ... and we see that all of these do satisfy 4x - 6y = 10.

Decreasing both x and y by the above values and starting with $\{4, 1\}$, we arrive at $\{-2, -3\}$, $\{-8, -7\}$, ... and we see that these also satisfy the equation 4x - 6y = 10.

So are these ..., $\{-8, -7\}$, $\{-2, -3\}$, $\{4, 1\}$, $\{10, 5\}$, $\{16, 9\}$, ... the solutions and does *x* indeed change by 6 and *y* by 4?

These are the solutions, but not all the solutions.

In this case when *x* and *y* increases (or decreases) by 3 and 2, we would still get integer solutions that satisfy the equation.

Increasing x and y by 3 and 2, starting from $\{4, 1\}$, we get $\{7, 3\}$, $\{10, 5\}$, $\{13, 8\}$, ... all of which satisfy the equation 4x - 6y = 10

Decreasing x and y by 3 and 2, starting from $\{4, 1\}$, we get $\{1, -1\}$, $\{-2, -3\}$, $\{-5, -5\}$, ... all of which satisfy the equation 4x - 6y = 10

Thus in successive solutions, ..., $\{1, -1\}$, $\{4, 1\}$, $\{7, 3\}$, ... we see that *x* and *y* changes not by the coefficients of other variables i.e. 6 and 4, but they change by 3 and 2. Why does this anomaly happen? Simply because the equation 4x - 6y = 10 was not in its lowest form. The equation on being reduced to its simplest form is 2x - 3y = 5 and obviously *x* and *y* have to change by 3 and 2 respectively.

Thus make sure you reduce the equation to its simplest form before proceeding to identify the quantum of changes in x and y in successive integer solutions.

4. Lastly to complete the generalisation, consider the equation 4x - 6y = 9. Firstly we check to see if the equation is in the most reducible form and having satisfied that it is we proceed. Banking on the above theory, we expect the value of *x* and *y* to change by 6 and 4 respectively. Since the integer values of *y* change by 4 in successive solution, we expect atleast one value from 0 to 3 to satisfy the equation. However on trying each of these values 0, 1, 2 and 3 successively we do not get any integer value for *x*.

Even if you try for any other values of y or x, you will not find any integer solution to this equation. The reason is that the coefficients of x and y are not co-prime and have a common

factor viz. 2. Thus the equation can be re-arranged to $2x - 3y = \frac{9}{2}$. If x and y are both integers,

the LHS is an integer and thus can never be equal to $\frac{9}{2}$. Thus this linear equation has no integer solution.


To sum up the above generalizations:

1. First get the equation in the lowest reducible form.

2. If x and y still have a common factor, after reducing it to the lowest reducible form, the equation will have no integer solutions

3. If x and y are co-prime in the lowest reducible form, then, in successive integer solutions to the equation, x and y will change by the coefficient of the other variable.

4. If the equation is of the type Ax + By = C, an increase in x will cause a decrease in y and if the equation is of the type Ax - By = C, an increase in x will cause an increase in y.

Identifying the first solution:

Consider the equation 5x + 9y = 103.

Since the coefficients of x and y are co-prime this equation would have infinite integer solutions. We have also learnt that in successive integer solutions when x increases (or decreases) by 9, y will decrease (or increase) by 5. But to use this to find the integer solutions we would have to first find one solution. The following text explains the efficient process of finding one solution to the equation. Since x increments by 9, one in every 9 consecutive integers would satisfy the equation. Thus even if we start plugging values of x successively as 0, 1, 2, ..., we would surely get a solution by the time we reach x = 8. But had we started with plugging values of y, we would have got an integer solution for y taking one value out of 0, 1, 2, 3 and 4. Thus to find the first solution, a time-saving technique would be to plug values for that variable which increments in steps.

If you start plugging values of x, starting from 0, you would reach an integer solution only for x = 6 i.e. 6 iterations in all. If you start plugging values of y, you would get an integer solution for y = 2 i.e. in the second iteration itself.

Counting the number of solutions

Consider the series 3, 16, 29, 42, ...

How many terms are there in the above series that are less than or equal to 800?

One would have to do such counting in numerous situations, one of them being finding the number of whole number solutions to Ax + By = C. Also the counting process is a very tedious one and also an error-prone area. Usually there is an error of ± 1 unless one is very careful. The following text explains a very efficient method to do such counting without committing any error.



Method 1: Writing the numbers in the form $d \times n + k$, where *d* is the common difference and *n* is natural number and *k* any integer. The number *n* is going to serve as our counting number and so should ideally begin from 1, but need not necessarily start with 1, as the two examples below show....

Since in the series 3, 16, 29, 42, ... the common difference is 13, the numbers should be thought as $13 \times 1 - 10$, $13 \times 2 - 10$, $13 \times 3 - 10$,

Dividing 800 by 13 we get the quotient to be 61 and a remainder of 7. Thus 793 is divisible by 13 and the number that is a part of the series is $13 \times 61 - 10$ i.e. 793 - 10 = 783. Since adding 13 to this number would not cross 800, we can safely say that $13 \times 62 - 10$ is also a number in the series and is the last such number (with the condition of being less than 800). Thus there are 62 such numbers in the series.

E.g. How many terms of 40, 57, 74, 91, ... are three digit natural numbers?

The numbers of the series can be written as 17n + 6.

The first three digit number of this type is with n = 6

Dividing 999 by 17, and finding the quotient as 58 and a remainder of 13 i.e. 986 is divisible by 17. Thus the last three digit number of this type is with n = 58

Thus there are 58 - 6 + 1 = 53 such numbers.

Method 2:

Taking the same example of finding the number of terms of 3, 16, 29, 42, ... that are less than or equal to 800?

Since the terms of the series 3, 16, 29, 42, ... differ by 13, we can say for sure that in any set of 13 consecutive numbers, there would be one and only one number that belongs to the series. Why?

If the first number of the set of 13 consecutive numbers belongs to the series, there would be no other number of the set belonging to the series. The next number in the series would be the number just greater than the last number of the set.

It is not possible for none of the numbers in a set of 13 consecutive numbers to not belong to the series, else there would be atleast two consecutive numbers of the series which differ by more than 13.

Thus if there is an A.P. with common difference *d*, in any set of *d* consecutive numbers, one and only one number of the set would belong to the A.P.

How do we use the above funda to find the number of terms on the series 3, 16, 29, 42, ... that are less than or equal to 800?



Consider sets of 13 numbers, starting with the first number of the given series. Thus in each of the sets 3-15, 16-28, 29-41, ... the first number of the set is a term in the given series. From 3 to 800 there are 800-3+1=798 numbers. These 798 numbers (from 3 to 600) can be formed in $\frac{798}{13} = 61\frac{5}{13}$ and thus from each of the 61 sets, the first one will be a part of the given series and since there is atleast one more number after forming complete sets of 13, the immediate next number will also be a part of the arrive.

immediate next number will also be a part of the series. In this method we just need to check if the division is complete or if there is atleast a remainder of 1. Thus it is slightly better off than the earlier.

E.g. How many terms of 40, 57, 74, 91, ... are three digit natural numbers?

The first three digit number of this form is $17 \times 6 + 6 = 108$. Starting with 108 and upto 999 i.e. in the set of 892 numbers, we can form $\frac{892}{17} = 52$ complete groups of 17 numbers and there would be 8 numbers leftover in an incomplete group.

By the above logic this implies there would be 53 numbers of the given series that are three digit numbers.

E.g. 10: How many natural number solutions exist for the equation 9x + 7y = 876

As x assumes 1, 2, 3, ..., 7, y would become 867, 858, 849, ... Checking which number in this series is divisible by 7, we can see that the first number in this series which is divisible by 7 is 840. Thus one solution to the equation is x = 4 and y = 120. Now x increases in steps of 7 and thus in successive whole number solutions, x will assume values 4, 11, 18, And obviously x has to be less than $\frac{876}{9} = 97.33$ or else y will become negative.

Considering the sets 4 - 10, 11 - 17, 18 - 24, ... from 4 to 97, we could have $\frac{97 - 4 + 1}{7} = \frac{94}{7} = 13\frac{3}{7}$. Since there is at least 1 more number than 13 complete sets,

there would be 14 natural number solutions to the equation.

If the question had been "how many natural number solutions exists for the equation 9x + 7y = 848?", would the answer change or be the same?

To finding the first solution, putting the value of *x* successively as 1, 2, 3, ..., we find 7y would be 839, not divisible by 7; 830, not divisible by 7; 821, not divisible by 7; 812 and this is divisible by 7. Thus first solution is for x = 4.

Now x will increase in steps of 7 and has to be less than $\frac{848}{9} = 94.22$



From 4 to 94, we could make $\frac{94-4+1}{7} = \frac{91}{7} = 13$ exact groups and thus there would be 13

solutions.

Many students erroneously use a technique that in such questions the answer is the integral part when the constant term divided by the product of the coefficients. This is not a fool proof method. It would give the correct answer but only in some cases e.g. in the question just solved, whole number solution to 9x + 7y = 848, it would give the correct answer because $\frac{848}{63} = 13.**$ and the answer is 13. But in the question we started with, whole number solution to 9x + 7y = 876, we would get a wrong answer by this technique as it would still give an answer of 13 because $\frac{876}{63} = 13.**$, but the correct answer is 14. So do not use this technique. It gets further complicated depending on the coefficient of *x* and *y* being factors of the constant term. So the

complicated depending on the coefficient of *x* and *y* being factors of the constant term. So the fool proof solution is to stick to the method of finding one solution and then proceeding.

E.g. 11: How many solutions exists for the equation 17x - 4y = 1 such that both x and y are whole numbers and y < 1000?

Make sure you start by plugging values of x as we have to get a solution for x being one of 1, 2, 3 or 4. The first solution is easily found to be x = 1 and y = 4. Now since the upper limit of y is given, let's work on the values that y could assume. Values of y would differ by 17 and thus for whole number solutions y would be 4, 21, 38, ...

From 4 to 999 (y is less than 1000), we could form $\frac{999-4+1}{17} = \frac{996}{17} = 58\frac{10}{17}$ groups.

Thus there are in all 59 whole number solutions to the equations.



Exercise

9. Find the number of natural number solutions and also the number of non-negative integer solutions to the following equations:

	i. $3a + 5b = 750$			
	(1) 250, 248	(2) 150, 148	(3) 51, 49	(4) 50, 48
	ii. $6a + 4b = 500$			
	(1) 21, 20	(2) 42, 41	(3) 42, 40	(4) 20, 19
	iii. $5a + 13b = 467$			
	(1) 6, 6	(2) 6, 5	(3) 7, 7	(4) 7, 6
10.	Find the number of who	ole number solutions to	the equation $7x - 9y = 1$	such that $y < 350$.
	(1) 49	(2) 50	(3) 51	(4) Infinite
11.	In CAT 2006, there were wrong answer was pena answer is a score of 50	re 75 questions. Each co alized 1 mark. In how m possible?	rrect answer was reward any different combinatio	ed by 4 marks and each ons of correct and wrong
	(1) 12	(2) 13	(3) 14	(4) Infinite
12.	How many integer solu	tions exist for the equation	ion $8x - 5y = 221$ such the	hat $x \times y < 0$.
	(1) None	(2) 5	(3) 6	(4) Infinite
13.	How many integer solution than 75?	tions exists for the equa	tion $11x + 15y = -1$ such	that both x and y are less
	(1) 5	(2) 7	(3) 12	(4) 13
14.	For how many positive integer solution.	integral values of N, les	ss than 40 does the equat	tion $3x - Ny = 5$ have no
	(1) None	(2) 13	(3) 11	(4) 8



Quadratic Equations

As already noted a quadratic equation is one in which the highest index of the variable is 2.

The general form of the quadratic equation is $ax^2 + bx + c = 0$.

Please note that *a* is the coefficient of x^2 , *b* is the coefficient of *x* and *c* is the constant term. DO NOT memorize them as the first coefficient is *a*, second is *b* and third is *c* because the order in which the equation is written can be changed. Also the coefficient has to be taken along with their signs. E.g. in the quadratic equation $4x - 10 + 3x^2 = 0$, a = 3 (and not 4), *b* is 4 and *c* is -10 (and not 10).

There are two ways to solve a quadratic equation, viz. method of factorization or use of formula. Please note that not all equations can be solved by the process of factorization and if the equation cannot be factorised, one would have to use the formula....

Roots of the quadratic equation $ax^2 + bx + c = 0$ is $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

E.g. 12: The sum of squares of three consecutive natural numbers is 302. What is the middle number?

Let the three numbers be x, (x + 1) and (x + 2).

Thus, $x^2 + (x + 1)^2 + (x + 2)^2 = 302$

 $x^{2} + x^{2} + 2x + 1 + x^{2} + 4x + 4 = 302$ i.e. $3x^{2} + 6x - 297 = 0$

 $x^{2} + 2x - 99 = 0$ i.e. (x + 11) (x - 9) = 0 i.e. x = -11 or 9.

Since *x* is a natural number, x = 9. Thus the middle number is 10.

Sum and Product of the roots:

To find the sum and product of roots, one need not find the roots and then add or multiply them. One can directly find the sum and product of the roots by just knowing the coefficients of the x^2 , x, and the constant term using the following formulae:

For the quadratic equation $ax^2 + bx + c = 0$,

Sum of the roots =
$$-\frac{b}{a}$$
 Product of the roots = $\frac{c}{a}$

The above should be obvious because we have been using the same funda while factorizing a quadratic expression. Just to refresh our memories, the quadratic $ax^2 + bx + c = 0$ could be re-arranged as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ and if *p* and *q* are the roots of this equation, then



$$x^{2} + \frac{b}{a}x + \frac{c}{a} = (x - p) \times (x - q) \qquad \Rightarrow x^{2} + \frac{b}{a}x + \frac{c}{a} = x^{2} - (p + q)x + pq$$

Equating the coefficients of both sides, $p + q = -\frac{b}{a}$ and $pq = \frac{c}{a}$

Most of the questions on quadratic equation in entrance exams are based on the above relations. So learn them thoroughly and also understand the following examples thoroughly.

E.g. 13: Find the sum of the roots of the equation $10 + 6x - 2x^2 = 0$

Using the formula, the sum of the roots is $-\frac{6}{-2} = 3$

E.g. 14: If one root of the equation $x^2 - 6x + c = 0$ is double the other root, find the value of *c*. Since one of the root is double of the other, we can assume the roots as *p* and 2*p*. Also from the given equation, we can know that the sum of the roots is 6. Thus p + 2p = 6 i.e. 3p = 6 i.e. p = 2.

Thus the two roots are 2 and 4

Now the product of the roots is *c* because a = 1. Thus $c = 2 \times 4 = 8$.

E.g. 15: If the roots of the equation $x^2 - (k+3)x + k = 0$ are reciprocals of each other, find the sum of the roots.

Since the roots of the equation are reciprocals of each other, the product of the roots is 1.

Thus
$$\frac{c}{a} = \frac{k}{1} = 1 \Longrightarrow k = 1$$

Sum of the roots = $-\frac{b}{a} = -\frac{-(k+3)}{1} = (k+3) = 4$

Forming the equation when the roots are given

If the roots of a quadratic equation are *p* and *q*, then the equation can be constructed as follows: $x^2 - (p+q)x + pq = 0$

In other words a quadratic equation can be constructed as follows:

 $x^2 - (\text{sum of the roots})x + \text{product of the roots} = 0$

E.g. 16: Find the equation whose roots are 3 and -7.

Sum of roots = 3 + (-7) = -4

Product of the roots = -21

Thus the equation is $x^2 - (-4)x + (-21) = 0$ i.e. $x^2 + 4x - 21 = 0$



E.g. 17: If *p* and *q* are the roots of the equation $x^2 - 2x + 3 = 0$, find the equation whose roots are (p + 3) and (q + 3).

In such questions it would not be the right strategy to find the root of the given equation because most probably the equation would not be factorisable and the roots would be imaginary or irrational. In this case the roots of the given equation, $p = 1 + \sqrt{2}i$ and $q = 1 - \sqrt{2}i$. To find the sum and product of (p + 3) and (q + 3) and then the equation would be a lengthy process.

The way out is as follows:

We need to find the equation: $x^2 - (p + 3 + q + 3)x + (p + 3)(q + 3) = 0$ i.e. $x^2 - (p + q + 6)x + (pq + 3p + 3q + 9) = 0$ Since *p* and *q* are the roots of $x^2 - 2x + 3 = 0$, we already know that p + q = 2 and pq = 3. Thus, the required equation is $x^2 - (2 + 6)x + (3 + 3 \times 2 + 9) = 0$ i.e. $x^2 - 8x + 18 = 0$

Determinant and Nature of the Roots:

The expression under the radical (root) sign i.e. $b^2 - 4ac$ is called the Determinant of the equation and is denoted as *D*. Thus $D = b^2 - 4ac$. The determinant of the equation is used to identify the Nature of the roots, as follows:

Determinant	Nature of Roots
D < 0 i.e. D is negative	Roots are imaginary
D = 0	Roots are real and equal
D > 0	Roots are real and unequal

E.g. 18: For what value of *b* will the equation $x^2 + bx - 9 = 0$ have equal roots?

Roots of a quadratic equation are equal when $b^2 - 4ac = 0$

Thus $b^2 - 4 \times 1 \times (-9) = 0$ i.e. $b^2 = 36$ i.e. $b = \pm 6$

Thus when b = 6 or -6, the equation will have equal roots.

Placement of the graph on X-Y plane

We have already seen that the roots of a quadratic polynomial are the points at which the graph of the polynomial function cuts the X-axis. Further, we have seen that depending on the value of D, the roots would be real and distinct, real and equal or imaginary. Depending on these conditions and depending on *a* being positive or negative, we could place the graph with respect to the X-axis in the following different ways:



	For positive <i>a</i>	For negative <i>a</i>
For D > 0		
For $D = 0$		
For D < 0		$\qquad \qquad $

We will look at these graphs once again when we learn inequalities and maxima and minima.

Relation between roots and coefficients for a higher degree equation

While it is very rarely asked to solve a cubic equation, there could be questions base on the roots of a cubic equation e.g. if p, q and r are roots of the equation $x^3 - 2x^2 + 5x - 4 = 0$, find the value of $p^2 + q^2 + c^2$. In such a question it is not necessary to find the roots of the equation and then compute the required value. It can be found directly by knowing the following relations between the roots of the equation and the coefficients:

If the roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are *p*, *q* and *r*, then we have

Sum of roots i.e. $p + q + r = -\frac{b}{a}$

Sum of roots taken two at a time i.e. $pq + qr + pr = \frac{c}{a}$

Sum of roots taken three at a time, in this case the product, i.e. $pqr = -\frac{d}{a}$

Using these relations for the question in the example, p + q + r = 2 and pq + qr + pr = 5. This is all we need to find the value of $p^2 + q^2 + c^2$ because of the algebraic identity

$$(p+q+r)^2 = p^2 + q^2 + c^2 + 2(pq+qr+pr)^2$$

Substituting the values, we have $4 = p^2 + q^2 + c^2 + 2 \times 5$ i.e. $p^2 + q^2 + c^2 = -6$

Do not be surprised that sum of three squares is a negative number. p, q and r could also be imaginary number and hence their squares could be negative.



The relation between roots and the coefficients could be generalized for any higher degree equation as well but the notation would become a trifle difficult for those who are not used to symbols of sigma (Σ). So we will just give the relation using an example of fourth degree equation and verbally point out how to generalize.

Consider the fourth degree equation $ax^4 + bx^3 + cx^2 + dx + e = 0$. Since it is a fourth degree polynomial, it would have four roots, say *p*, *q*, *r* and *s*. The relation between the roots and the coefficients can be remembered by sequentially considering 'sum of roots taken 1 at a time', 'sum of roots taken 2 at a time', 'sum of roots taken 3 at a time' and so on. The right hand side

of the relations will sequentially be $-\frac{b}{a}, \frac{c}{a}, -\frac{d}{a}, \frac{e}{a}, \dots$ Note the alternating minus and plus

sign, starting with a minus sign. If this is not very clear, see how the rule is applied to this fourth degree equation:

Sum of roots taken 1 at a time, i.e. sum of roots $p + q + r + s = -\frac{b}{a}$

Sum of roots taken 2 at a time, i.e. $pq + pr + ps + qr + qs + rs = \frac{c}{a}$

Sum of roots taken 2 at a time means sum of all possible pairs of roots

Sum of roots taken 3 at a time, i.e. $pqr + pqs + prs + qrs = -\frac{d}{a}$

Sum of roots taken 3 at a time means sum of all possible combinations of three roots taken at a time.

Sum of roots taken 4 (all in this case) at a time, i.e. product of roots, $pqrs = \frac{e}{a}$

In the above, the most important relations are the sum of roots and the product of roots, so let's look at them once again carefully...

For any polynomial of degree n, $ax^{n} + bx^{n-1} + \dots + k$, where k is the constant term,

sum of roots =
$$-\frac{b}{a}$$
 product of roots = $(-1)^n \times \frac{k}{a}$

Thus if the question asked is what is the sum and product of all the 100 roots of the polynomial $2x^{100} - 5 = 0$, the answer should be evident as follows:

Sum of the roots = 0, because the coefficient of x^{99} is zero as it is absent and

Product of the roots = $(-1)^{100} \times \frac{-5}{2} = -\frac{5}{2}$



Exercise

- 15. A quantity of rice at a certain rate costs Rs. 40. If the quantity purchased was lesser by 6 kg and the rate was higher by 1.5 Rs/kg, the cost would still be Rs. 40. What is the original rate of rice in Rs/kg?
 - (1) 2.5 (2) 4 (3) 10 (4) 16
- 16. Solve for x; $x^{\frac{2}{3}} 4x^{\frac{1}{3}} 12 = 0$ (1) -2, 6 (2) 6 (3) 216 (4) -8, 216
- 17. Find the value of p in $x^2 + px + 8 = 0$ for each of the following two cases:
 - i. If one root is square of the other root.
 - (1) 6 (2) -6 (3) 6 or -6 (4) $2 + \sqrt{2}$
 - ii. If both the roots are equal
 - (1) $\pm 4\sqrt{2}$ (2) $\pm 2\sqrt{2}$ (3) $\pm \sqrt{2}$ (4) $\pm 8\sqrt{2}$
 - iii. If the difference of the two roots is 2.
 - (1) 6 (2) -6 (3) 6 or -6 (4) $2 + \sqrt{2}$

18. If *p* and *q* are the roots of $x^2 - 7x - 6 = 0$, find the value of $\frac{p}{q} + \frac{q}{p}$

- $(1) \ 61/6 \qquad (2) 61/6 \qquad (3) \ 73/6 \qquad (4) 73/6$
- 19. If 3p + 1 and 3q + 1 are the roots of the equation $x^2 7x + 10 = 0$, find the equation whose roots are *p* and *q*.

(1) $x^2 - 23x + 112 = 0$ (2) $x^2 + 23x + 112 = 0$ (3) $9x^2 + 15x + 4 = 0$ (4) $9x^2 - 15x + 4 = 0$

20. Iterative patterns reducible to quadratic form:





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(1) 1.5 (2) 1.75 (3) 1.833 (4) 2

21. Find the value of p if the equation $3x^2 - 2x + p = 0$ and $6x^2 - 17x + 12 = 0$ have a common root.

- (1) 15/4 (2) 8/3 (3) 8/15 (4) Both (1) and (2)
- 22. If the quadratic equations $ax^2 + 2cx + b = 0$ and $ax^2 + 2bx + c = 0$ has exactly one common root, then find the value of a + 4(b + c).
 - (1) -2 (2) -1 (3) 0 (4) 1

23. If *p*, *q* and *r* are the roots of the equation $x^3 - 3x^2 + x + 1 = 0$, find the value of $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$

- (1) 3 (2) 1 (3) -1 (4) -3
- 24. If *p*, *q* and *r* are the roots of the equation $2x^3 + 4x^2 3x 1 = 0$, find the value of $(1-p) \times (1-q) \times (1-r)$
 - (1) 1 (2) 2 (3) 3 (4) 4





The Basic Symbols and Simultaneous Conditions:

Inequalities come into play when two quantities or expressions are not equal. In such case one of them is less than the other or the other is more than the first one. To denote such relations we use symbols < (less than), > (greater than), \leq (less than or equal to), \geq (greater than or equal to).

In finding the solution set of inequalities, it is often very useful to view the inequalities on the number line. The following will help you visual the various situations encountered. Make sure you understand the OR and AND situations thoroughly before moving on. In all the following figures, the bold line refers to the acceptable values.

1.
$$x > 5$$
 \leftarrow

The light circle means that the value 5 is not acceptable

2.
$$x \ge 5$$

The dark circle means that the value 5 is acceptable

3.
$$x > -3$$

As seen in the above three examples, greater than can always be thought of as "to the right of". This is very helpful in visualising without the need of drawing a number line.



As seen in the above two examples, less than can always be thought of as "to the left of". This is very helpful in visualising without the need of drawing a number line.



6. The case of *and*: x > 2 and x < 8

x lies to the right of 2 *and* to the left of 8.

Since this is a range bound region, i.e. it is bound between two points, the two inequalities are combined into one single relation as $2 \le x \le 8$ and is read as 2 is less than *x* and *x* is less than 8.

When you write such a relation, a good practice is to write it as given above and not as 8 > x > 2. While this is also technically write because 8 is greater than *x* and *x* is greater than 2, but we do not write it this way. We always write it as 2 < x < 8 because on the number line 2 comes to the left of 8.

Other ways of writing the same relation is (2, 8). Curved brackets are used to denote exclusion, whereas square brackets are used to denote inclusion as shown in below example.

7. $x \ge -5$ and x < 4



Note that the value of -5 is acceptable but 4 is not acceptable as denoted by the dark circle and the light circle respectively.

Again since it is a range-bound solution, we express the inequalities in one single relation as $-5 \le x \le 4$. Another way of writing the same relation is [-5, 4)

8. Comparison of and and or

When two conditions are joined with an *and*, both the conditions have to be satisfied. Thus, as seen above in the case of x > 2 and x < 8, x can accept only those values that are greater than 2 and are also less than 8.

But when two conditions are joined with an *or*, either of the condition has to be satisfied. Thus if we have to plot the acceptable values of *x* that satisfy x > 2 or x < 8, it will be the entire number line because any point on the number line will be either be greater than 2 or will be less than 8.

These values are less than 8 and hence acceptable These values are less than 8 and hence acceptable These values are conditions these values are conditions these values are hence acceptable

x > 2 or $x < 8 \implies$ any one condition, > 2 or < 8 has to be satisfied



One does not get conditions exactly like the above, but one does get conditions with *or* that result into more specific answers. So you should understand *or* and *and* carefully. A few comparisons between *and* and *or* are given below:



9. The disjoint set: x < 2 or x > 8

As seen in the last of the earlier examples, we could also have a solution of two disjoint sets:

 $x < 2 \text{ or } x > 8 \implies$ any one condition has to be satisfied





Be careful not to write this relation as 2 > x > 8. Each of the two inequality in 2 > x > 8 are individually correct because 2 > x and x > 8 are true according to the given relation. But when you club them together, as in 2 > x > 8, it implies 2 > x and x > 8 and this is not same as the conditions given. Infact 2 > x and x > 8 does not have any solution because no value is less than 2 *and* greater than 8 simultaneously.

Thus the above solution has to written with an *or* only, as x < 2 or x > 8, and do not try to club them in one single relation.

The above relation is sometimes written as $(-\infty, 2) \cup (8, \infty)$, where the symbol \cup refers to 'union' i.e. *or*.

10: We could also have any combination of the relations e.g. (x < 8 and x > 2) or $x \le 0$.

The above relation would translate to a solution set equivalent to:



While writing this could also be written as $x \le 0$ or 2 < x < 8.

Another way to write would be: $(-\infty, 0] \cup (2, 8)$

11: A combination of relations which we will encounter very often is of the type:

If x < 0 then x > -5 or if $x \ge 0$, then $x \le 4$

Taking each condition at a time, this relation can be thought as:

x < 0 and x > -5 i.e. -5 < x < 0 or $x \ge 0$ and $x \le 4$ i.e. $0 \le x \le 4$

i.e. -5 < x < 0 or $0 \le x \le 4$ i.e. $-5 < x \le 4$

On a number line the solution set will be:

A couple of more relations of this type are as follows:

If $x \le 5$ then x > -2 or if x > 5 then x > 8

Taking each condition at a time, this relation can be thought as:

 $x \le 5$ and x > -2 i.e. $-2 < x \le 5$ or x > 5 and x > 8 i.e. x > 8i.e. $-2 < x \le 5 \cup x > 8$



If x < 2 then x < -2 or if $x \ge 2$ then x > 5Taking each condition at a time, this relation can be thought as: x < 2 and x < -2 i.e. x < -2 or $x \ge 2$ and x > 5 i.e. x > 5i.e. $x < -2 \cup x > 5$ (-1) + (-2) +

Operations on Inequalities

Having understood the use of inequality symbols and also how to satisfy two simultaneous conditions, the next agenda is to learn how one can manipulate an expression including an inequality.

Inequalities behave differently than equalities. We will first deal with linear inequalities, have a look at the common errors that most novices make and then move on to higher degree inequalities.

The following table compares the operations that are valid for equalities but may or may not be valid for inequalities.

	Operation done	For the equality	For the inequality
		a = b	a > b
1.	Adding <i>k</i> to both sides or	$a \pm k = b \pm k$	$a \pm k > b \pm k$
	Subtracting k from both sides		The inequality remains same for k
			being positive or negative
2.	Multiplying <i>k</i> to both sides	$a \times k = b \times k$	If k is positive, inequality remains same
	Dividing both sides by k	$\frac{a}{k} = \frac{b}{k}$	i.e. $a \times k > b \times k$ and $\frac{a}{k} > \frac{b}{k}$
			If k is negative, inequality reverses i.e.
			$a \times k < b \times k$ and $\frac{a}{k} < \frac{b}{k}$
3.	Taking reciprocals	$\frac{1}{a} = \frac{1}{b}$	If both <i>a</i> and <i>b</i> are of same sign,
			inequality reverses i.e. $\frac{1}{a} < \frac{1}{b}$
			If <i>a</i> is positive and <i>b</i> is negative,
			inequality remains same i.e. $\frac{1}{a} > \frac{1}{b}$



	Operation done	For the equality	For the inequality
		a = b	a > b
4.	Squaring both sides	$a^2 = b^2$	Only if both <i>a</i> and <i>b</i> are positive, would inequality remain same i.e. $a^2 > b^2$ If both <i>a</i> and <i>b</i> are negative, inequality would reverse i.e. $a^2 < b^2$
			If <i>a</i> is positive and <i>b</i> is negative, nothing can be said, would depend on the magnitudes.
5.	Cubing both sides	$a^3 = b^3$	$a^3 > b^3$, for all values of $a > b$

Let's take each of these turn by turn and apply them to scenarios of solving linear inequalities. Also focus on the common errors that occur.

1. Adding or subtracting both sides of an inequality

E.g. 19: Consider solving 3x - 8 > 2x + 1

When we transpose –8 to the RHS what we are infact doing is adding 8 to both sides. Since adding same value to both sides does not affect the inequality, we can do the operation without too much worry

Thus, 3x > 2x + 9

Similarly transposing 2*x* to the LHS is also perfectly possible, even though we do not know the sign of 2*x*, because in doing so we are subtracting 2*x* from both sides and this can be done without affecting the inequality sign for all positive and negative values. Thus, 3x - 2x > 9 i.e. x > 9.

2. Multiplying both sides of an inequality

This is an area where a lot of errors happen. If $\frac{a}{b} > 4$, can we say that a > 4b? Not really,

because in this case we have multiplied both sides of the inequality by b and we do not know if b is positive or negative. Thus we cannot ascertain the inequality sign between a and 4b.

The only way to proceed is to consider two cases, *b* being positive or negative and then taking a > 4b and a < 4b respectively.

E.g. 20: Solve for *x*;
$$\frac{3x-5}{x} > 2$$
 if $x \neq 0$

If x is negative i.e. x < 0, then we have 3x - 5 < 2x i.e. x < 5

This is equivalent to saying x < 0 and x < 5 and we have learnt that the values that satisfy both these conditions are x < 0.



If x is positive i.e. x > 0, we have 3x - 5 > 2x i.e. x > 5 i.e. x > 0 and x > 5 i.e. x > 5

Now *x* could be negative or positive and thus we have x < 0 or x > 5

Thus the solution set for
$$\frac{3x-5}{x} > 2$$
 is $x < 0$ or $x > 5$.

We will see a more efficient method while dealing with higher degree inequalities.

Another common error of this nature is while solving inequalities of the type $\frac{1}{r} > \frac{1}{2}$. By cross-

multiplying we get 2 > x i.e. x < 2. Is this the solution to $\frac{1}{x} > \frac{1}{2}$? No, because x = -1 is a part

of the solution so arrived at but it does not satisfy the given inequality i.e. $\frac{1}{x} > \frac{1}{2}$. The error

committed is same as explained earlier that we have multiplied both sides of the inequality by *x*, the sign of which we do not know. The correct working is: If *x* is positive, then x < 2 i.e. a solution 0 < x < 2. If *x* is negative, then x > 2 i.e. no solution. One could also arrive at this solution by logically thinking that no negative values can satisfy the inequality. So the final solution is 0 < x < 2.

3. Dividing both sides of an inequality

Again an area where a lot of errors occur.

E.g. 21: Solve for *x*; $x^2 < 4x$.

Can we cancel out *x* from both sides and say that x < 4 is a solution? Consider x = -1, which is a part of the solution so arrived at and hence should satisfy the inequality. But it does not since $(-1)^2$ is not less than $4 \times (-1)$. So obviously x < 4 is not a correct solution set.

What went wrong is that when you cancel out x from both sides, mathematically we have divided both sides by x and when we divide both sides of an inequality we ought to be careful about the sign of the number we are dividing with because it affects the inequality sign. But we do not know if x is positive or negative. Hence we cannot cancel out x from both sides.

One way out is again considering two cases, x being positive or negative.

If x is negative, x < 0, dividing both sides by x (a negative quantity), we get x > 4. Thus, x < 0 and x > 4 i.e. no solution.

If x is positive, x > 0, then dividing both sides by x (a positive quantity), we get x < 4.

Thus, x > 0 and x < 4 i.e. 0 < x < 4

So the final solution is 0 < x < 4.



Again we will see a very simpler solution for this while dealing with higher degree inequality. In this part just focus on the errors that you have to avoid.

4. Squaring or taking the square root

Consider solving $x^2 < 4$. Taking the square root gives us x < 2. Again this solution would be wrong because x = -3, part of the solution so arrived, does not satisfy the inequality $x^2 < 4$. And this should have been expected knowing that squares of negative numbers are positive. The correct solution to this inequality will be -2 < x < 2.

So given an inequality, we cannot square both sides or take the square root of both sides. Doing so may give us a wrong solution.

Higher Degree Inequalities

E.g. 22: Solve for *x*; $x^2 - x - 6 < 0$

Since we have learnt the quadratic polynomial function in details, one should be able to answer this even without learning any new process. One just needs to refresh the graph of the polynomial function. Now since $x^2 - x - 6$ can be factorised as $(x - 3) \times (x + 2)$, the graph could be plotted as:



Now when we have to solve $x^2 - x - 6 < 0$, we have to find those values of x for which the expression $x^2 - x - 6$ is negative. Form the graph, it is obvious that for all values of x that lie between -2 and 3, and for no other value would it be negative. Thus the solution to the inequality $x^2 - x - 6 < 0$ would be -2 < x < 3

It should not be too much of a difficulty in comprehending that had the inequality been $x^2 - x - 6 > 0$ the solution set would be x < -2 or x > 3.



While the above should be the approach to solve straight forward quadratic inequalities, one should also learn the below mentioned technique because it helps in numerous situations where the graph of the function is not known...

$$x^2 - x - 6 < 0$$
 i.e. $(x - 3) \times (x + 2) < 0$

Consider the factor (x - 3). For x = 3, the factor will be equal to zero; for all x < 3 (for x lying on the left of 3), the factor will be negative; for all x > 3 (x lying on the right of 3), the factor will be positive.

Similarly, for the factor (x + 2), we can say that: for x = -2, it will be zero; for x < -2 (*x* lying on the left of -2), the factor is negative; for x > -2 (*x* lying on the right of -2), the factor is positive. Thus, we divide the entire range of x ($-\infty$ to ∞ , the number line) into different regions by plotting the points where each of the factor becomes equal to zero as shown below. Then in each of the region we can ascertain the polarity (positive or negative) of each factor and also of the product of the factors...

		0	
	-2	· · · · · · · · · · · ·	3
For x in region	x < 2	2 < x < 3	x > 3
Polarity of $(x - 3)$	-ve	-ve	+ve
Polarity of $(x + 2)$	-ve	+ve	+ve
Polarity of product $(x-3) \times (x+2)$	+ve	-ve	+ve

This method is important to be known because it is very useful to solve inequalities for which we cannot plot the graph easily...

E.g. 23: Solve for *x*;
$$\frac{(x-3)}{(x+2)} > 0$$

Obviously we cannot multiply by (x + 2) unless we take two cases, once considering it positive and once considering it negative. We have seen earlier that such a solution is a little cumbersome because you have to handle multiple inequality relations linked with an *and* or an *or*. A more efficient way would be to consider the polarity of each of the factor viz. (x - 3) and (x + 2) in different regions and accordingly find the polarity of the ratio in different regions. In this case the polarity of each of the factor (x - 3) and (x + 2) would obviously be the same as just seen in the above diagram and thus the polarity of the ratio $\frac{(x-3)}{(x+2)}$ will also be the same as that of $(x - 3) \times (x + 2)$



4	(·····································	0 .	
	-2		3
For x in region	x < 2	2 < x < 3	<i>x</i> > 3
Polarity of $(x - 3)$	-ve	-ve	+ve
Polarity of $(x + 2)$	-ve	+ve	+ve
Polarity of ratio $\frac{(x-3)}{(x+2)}$	$\frac{)}{)}$ +ve	-ve	+ve

From the above figure, it should be obvious that $\frac{(x-3)}{(x+2)}$ will be positive only for x < -2 or x > 3.

In fact, one should realize that the solution of each of the following is the same:

a.
$$(x-3) \times (x+2) > 0$$
 b. $\frac{(x-3)}{(x+2)} > 0$ c. $\frac{(x+2)}{(x-3)} > 0$ d. $\frac{1}{(x-3) \times (x+2)} > 0$

The above method can also be extended for a cubic or higher degree inequality...

E.g. 24: Solve for x; $x^3 - 5x^2 - x + 5 \le 0$

The first step would be to factorise the polynomial function. A glance at the polynomial function should immediately make it obvious that when x = 1, the polynomial will be equal to zero. Thus (x - 1) is a factor.

Similarly, if one is very observant, one can also realize that even for x = -1, the polynomial becomes zero. Thus (x + 1) is also a factor. Now it is a small step to identify the third factor by comparing the terms in x^3 and the constant terms on the two sides of $x^3 - 5x^2 - x + 5 = (x - 1) \times (x + 1) \times (ax + b)$. Since coefficient of $x^3 = 1$, hence $1 \times 1 \times a = 1$ i.e. a = 1 and since constant = 5, $(-1) \times 1 \times b = 5$ i.e. b = -5. Thus the third factor is (x - 5).

Alternately, if one did not realize that (x + 1) is also a factor, one could have continued with the factorization as learnt earlier...

$$x^{3}-5x^{2}-x+5=(x-1)\times(ax^{2}+bx+c)$$

Equating coefficient of x^3 , $1 \times a = 1$ i.e. a = 1

Equating constant term, $(-1) \times c = 5$ i.e. c = -5.

Hence, $x^3 - 5x^2 - x + 5 = (x - 1) \times (x^2 + bx - 5)$

Equating coefficient of x^2 , $1 \times b + (-1) \times 1 = -5$ i.e. b = -4

Thus, the second factor can be found to be $x^2 - 4x - 5$ which can be factorised further as $(x - 5) \times (x + 1)$



This factorization process is learnt earlier and is again explained in details here just for refreshing the memory. Here onwards any polynomial function will be factorised directly in one step, assuming you are thorough with the process now.

Now the inequality to be solved is $(x - 1) \times (x + 1) \times (x - 5) < 0$

Each of the factors will become equal to zero at 1, -1 and 5 respectively. Plotting these points on the number line and knowing that for all *x* lying on the left of each respective points, the factor will be negative and for all *x* lying on the right of each respective points, the factor will be positive, we can visualize the polarity map as follows:

	~	0		
	A	-1 1	1	5
For x in region	x < -1	-1 < x < 1	1 < x < 5	x > 5
Polarity of $(x - 1)$	-ve	-ve	+ve	+ve
Polarity of $(x + 1)$	-ve	+ve	+ve	+ve
Polarity of $(x - 5)$	-ve	-ve	-ve	+ve
Polarity of the product $(x-1) \times (x+1) \times (x-5)$	-ve	+ve	-ve	+ve

Thus the product $(x - 1) \times (x + 1) \times (x + 5)$ would be negative when x < -1 or 1 < x < 5.

Having learnt the approach to tackle product or ratios of two or more than two factors of the type $(x \pm a)$, let us learn a few faster techniques to tackle such questions such that we avoid drawing the above figure...

1. Start from the right-most region

When faced with solving any inequality like $(x - 1) \times (2x + 3) \times (3x + 10) < 0$, one should just plot the points where each factor becomes zero, like...

$$\begin{array}{c}
0 \\
\hline
-1/2 \\
1 \\
10/3
\end{array}$$

Having done this, it is prudent to start identifying the polarity of the product $(x - 1) \times (2x + 3) \times (3x + 10)$ in the right-most region. The reason to start from the rightmost region is that in this region, the values of *x* are to the right of each of the plotted points and hence each of the factors is positive. When all the factors are positive, the product (or the ratio) has to be positive. Thus, we see that in the right most region, the required product will always be positive (see solved example 25 below for a more clearer understanding of 'always').

Now as we move leftwards, the polarity of the product (or ratio) will be alternately negative and positive in the regions, like...





Polarities of product or any ratio of (x - 1), (2x + 3) and (3x + 10)In right-most region will always be positive, and moving leftwards will alternate between negative and positive

If you are using the above technique, make sure that each factor is expressed in the form of $(ax \pm b)$ i.e. no factor should have the coefficient of *x* as negative e.g. the above method would not work if any factor is of the type (3 - x) or (3 - 2x). Consider (3 - x). It will be zero for x = 3. When *x* assumes any value greater than 3, i.e. *x* lies to the right of 3, the factor (3 - x) will be negative. But in the above technique we have used the fact that whenever *x* lies to the right of any point, the factor relating to that point will be positive. So what do we do when there exists a factor with the coefficient of *x* being negative? See the following example for an easy solution...

E.g. 25: Solve for $x: 4 + 3x - x^2 \le 0$

On factorizing, $(1 + x) \times (4 - x) \le 0$

Multiplying by -1, we get $(1 + x) \times (x - 4) \ge 0$. Note that the inequality sign reverses since we have multiplied with a negative number.

This product is of the standard type with coefficient of x being positive and thus can be solved in the regular manner...



Since we need $(1 + x) \times (4 - x) \le 0$ i.e. $(x + 1) \times (x + 4) \ge 0$, the solution will be $x \le -1$ or $x \ge 4$.

Another point worth noting is that in this example the product could also be equal to zero and hence the solution also includes the specific points as meant by the use of less than or equal to symbol.



2. It does not matter if it is a product or a ratio, well almost

Consider the product $(x - 1) \times (x + 1) \times (x - 5)$. Each of the factor becomes zero when x = 1, x = -1 and x = 5 respectively. When x > 5, x lies to the right of each of the three points and thus each of (x - 1), (x + 1) and (x - 5) is positive.

Now whether we are looking at the product $(x - 1) \times (x + 1) \times (x - 5)$ or any ratio of these

factors e.g.
$$\frac{(x-1)}{(x+5)\times(x+1)}$$
 or $\frac{(x-1)\times(x+5)}{(x+1)}$ or $\frac{1}{(x-1)\times(x+5)\times(x+1)}$, all of them will be

positive.

The fact that each of the above expressions would have the same polarity (positive or negative) is valid for any *x*. Consider *x* to be between 1 and 5. Thus since *x* is to the right of 1 and also to the right of -1, the factors (x - 1) and (x + 1) will be positive and the other factor (x - 5) will be negative. With these polarities you can again check that the product or any of the above ratios, are all negative. Thus it hardly matters if you are solving for $(x - 1) \times (x + 1) \times (x - 5) < 0$ or for

$$\frac{1}{(x-1)\times(x+5)\times(x+1))} < 0 \text{ or for } \frac{(x-1)}{(x+5)\times(x+1))} < 0.$$
 The solution for each of these will be

the same. However if the inequality sign was not a strict inequality but also included equal to, then matters would be slightly different...

The solution for
$$(x-1) \times (x+1) \times (x-5) \le 0$$
 would be $x \le -1$ or $1 \le x \le 5$

But the solution set for
$$\frac{(x-1)}{(x+5)\times(x+1)} \le 0$$
 would be $x \le -1$ or $1 < x < 5$

This is because while there is a less than or equal to symbol, the factors (x + 5) and (x + 1) cannot be equal to zero as they are in the denominator and thus it would make the ratio undefined.

Thus the solution would differ for a product and a ratio only if the inequality symbol includes 'equal to' as well and in this case also, the difference in the solution, in case of a product or a ratio, would be just in the inclusion or exclusion of the points where factors are zero. The regions would be the same though.

3. When there are powers of certain factors.

E.g. 26: Solve for *x*; $(x-3)^2 \times (x-2) \times (x+2) \ge 0$

Whatever be the polarity of (x - 3), the polarity of $(x - 3)^2$ will always be positive. Thus for the entire expression to be positive, $(x - 2) \times (x + 2)$ has to be positive i.e. $(x - 2) \times (x + 2) \ge 0$. The solution for which is a straight-forward $x \le -2$ or $x \ge 2$.

Thus from any expression of the type $(x - 3)^2 \times (x - 2) \times (x + 2)$, you can safely ignore any squares or for that matter any even powers as they will always be positive. Can you ignore it completely? Not really...



Had the above question been $(x-3)^2 \times (x-2) \times (x+2) > 0$, the solution x < -2 or x > 2 would have been correct, but for the fact that *x* cannot be equal to 3, the point corresponding to the ignored factor (x-3) becoming zero. Because if x = 3, the expression in the question would become equal to zero whereas the question requires the expression to be strictly less than zero. Thus the correct solution would be x < -2 or x > 2 but $x \neq 3$.

E.g. 27: Solve for *x*; $(x-3)^3 \times (x-2) \times (x+2) < 0$

Since we are interested only in the polarity of $(x - 3)^3$, and since it would be the same as that of (x - 3) for any value of x, we can just consider the above question to be $(x - 3) \times (x - 2) \times (x + 2) < 0$ and solve for this expression. Since the expression has to be negative, it will be so in the second and fourth region from the left end i.e. 2 < x < 3 or x < -2.

- 4. When a certain quadratic expression does not have real roots.
- **E.g. 28:** Solve for *x*; $(x 3) \times (x^2 x + 1) > 0$

When a quadratic expression cannot be factorised, check if the expression has real roots before proceeding to find the roots using the formula. In this expression $(x^2 - x + 1)$ would have imaginary roots as its determinant, $b^2 - 4ac$ is negative. In such case remember that the graph of the polynomial function with respect to the X-axis would be as follows...



Thus for positive *a*, the expression assumes only positive values for all values of *x*. Thus $(x^2 - x + 1)$ would always be positive for all real values of *x* and so the question just boils down to solving (x - 3) > 0 for which the solution is x > 3.



5. If the RHS is a number other than zero

So far we were solving just for the given expression to be positive or negative. When the (r + 4)

question is of the type
$$\frac{(x+4)}{(x+1)} > 3$$
, it is not enough to just know the polarity of $(x+4)$ and $(x+1)$

for various values of x. In such cases, rather than considering two cases, (x + 1) being positive and negative and then cross-multiplying, a easier solution would be to transpose the number on the RHS to the LHS...

E.g. 29: Solve for x;
$$\frac{(x+4)}{(x+1)} > 3$$

$$\Rightarrow \frac{(x+4)}{(x+1)} - 3 > 0 \Rightarrow \frac{(-2x+1)}{(x+1)} > 0 \Rightarrow \frac{(2x-1)}{(x+1)} < 0$$
i.e. solution would lie between the two point, $-1 < x < \frac{1}{2}$

Having learnt all the techniques and the variations in problems, let's close the loose ends that we had left when earlier in the text it was written that later we will learn more efficient ways to solve certain situations...

E.g. 30: Solve for *x*;
$$\frac{3x-5}{x} > 2$$
 if $x \neq 0$. (Same as E.g. 20)

Instead of cross-multiplying as was done when this example was first solved (earlier in this text), let's take 2 to the LHS...

$$\frac{3x-5}{x} - 2 > 0 \quad \Rightarrow \frac{x-5}{x} > 0 \quad \text{i.e. } x < 0 \text{ or } x > 5.$$

E.g. 31: Solve for *x*; $\frac{1}{x} > \frac{1}{2}$

$$\frac{1}{x} > \frac{1}{2} \Longrightarrow \frac{1}{x} - \frac{1}{2} > 0 \qquad \Longrightarrow \frac{2-x}{2x} > 0 \implies \frac{x-2}{2x} < 0 \implies 0 < x < 2$$

E.g. 32: Solve for $x^2 < 4x$. (Same as E.g. 21)

 $x^2 - 4x < 0$ i.e. x(x - 4) < 0 i.e. 0 < x < 4

And lastly, though this can be done by common sense..

E.g. 33: Solve for *x*; $x^2 < 4$

$$x^2 - 4 < 0$$
 i.e. $(x + 2) \times (x - 2) < 0$ i.e. $-2 < x < 2$



60 | Inequalities

Exercise

1.	Find the number of int	egral values of x that wi	Il satisfy $\frac{1}{x} \le -\frac{1}{3}$	
	(1) 4	(2) 3	(3) 2	(4) Infinite
2.	Find the number of int	egral values of x that wi	Il NOT satisfy $-\frac{1}{3} < \frac{1}{x} <$	$-\frac{1}{8}$
	(1) 10	(2) 11	(3) 12	(4) Infinte
3.	How many positive int	egral values of x will sa	tisfy the relation $x^3 - 9x$	$z \ge 7x$
	(1) 4	(2) 8	(3) 9	(4) Infinite
4.	If $-24 \le x \le 12$ and -6	$5 \le y \le -3$, find the maximum field $y \le -3$, find the maximum field $y \le -3$.	mum and minimum valu	ue of $\frac{x}{y}$
	(1) 8, -2	(2) 4, -2	(3) 8, -4	(4) $\infty, -\infty$
5.	If $-12 \le x \le 12$ and -6	$5 \le y \le 3$, find the maxim	num and minimum value	e of $\frac{x}{y}$
	(1) -2, -4	(2) 4, -2	(3) 4, -4	(4) $\infty, -\infty$
6.	If p , q , r and s are the r where $-7 \le x \le -3; -7$	number of distinct integr $\leq y \leq 3$, then which of t	ral values that x^2 , y^2 , x^3 as the following statements	nd <i>y</i> ³ can respectively assume, is true?
	(1) $p = q = 41$ and $r = 2$	s = 361	(2) $p = q = 41, r = 317$	and $s = 361$
	(3) $p = 41, q = 9, r = 3$	17 and $s = 361$	(4) None of the above.	

Directions for questions 7 to 18: Find the number of integral values of *x* in the range $-10 \le x \le 10$ that satisfy each of the following conditions:

7.	$(2x+1)^2 > 36$			
	(1) 8	(2) 14	(3) 15	(4) 7
8.	$x^2 - 10x - 24 \le 0$			
	(1) 15	(2) 14	(3) 13	(4) 12
9.	$\frac{1}{x-2} \le 2$			
	(1) 21	(2) 20	(3) 1	(4) 0



10.	$2-x-x^2 \ge 0$			
	(1) 4	(2) 3	(3) 2	(4) 17
11.	$(x-1)^2 \times (x-2)^3 \times (x-2)^3$	(-3) > 0		
	(1) 21	(2) 20	(3) 18	(4) 0
12.	$\frac{(x-1)^2(x-3)}{(x-2)^3} \le 0$			
	(1) 0	(2) 1	(3) 2	(4) 3
13.	$5 < 3x - x^2$			
	(1) 0	(2) 4	(3) 11	(4) 21
14.	$\frac{x^2 - 5x + 6}{x^2 + x + 1} < 0$			
	(1) 0	(2) 1	(3) 2	(4) 21
15.	$\frac{x^2 + 2x - 3}{2x - 1 - x^2} \le 0$			
	(1) 17	(2) 18	(3) 4	(4) 3
16.	$\frac{x^2 + 6x - 7}{x^2 + 1} \le 2$			
	(1) 0	(2) 1	(3) 20	(4) 21
17.	$\frac{x+1}{\left(x-1\right)^2} < 1$			
	(1) 7	(2) 8	(3) 17	(4) 4
18.	$2x^3 + x^2 - 25x + 12 > 0$			
	(1) 4	(2) 7	(3) 11	(4) 15
19.	How many integral val	ues of x satisfy the inequ	uality: $x^4 - 29x^2 + 100 \le$	0
	(1) 11	(2) 8	(3) 6	(4) 4
20.	How many integral val	ues of x satisfy the inequ	uality: $x - 5x^{\frac{1}{2}} - 6 \le 0$	
	(1) 37	(2) 36	(3) 6	(4) 8





The Absolute Value of any number, x, is denoted as |x| and is read as 'Mod of x' or 'Mod x'.

Absolute Value implies the 'magnitude' only, and is irrespective of the sign. Thus,

$$|3| = 3$$
, $|-2| = 2$, $|1.5| = 1.5$, $|-0.7| = 0.7$

Thus, we see that when the number is positive, its absolute value is the same number. But when the number is negative, the absolute value is not the number but only its magnitude. To arrive at the magnitude of the negative number, we can simply negate the number i.e. take the negative of the negative number.

Thus, we arrive at the formal definition of Absolute Value as follows:

|x| = x, if x is positive or zero = -x, if x is negative.

A common difficulty faced by beginners is to comprehend how can |x| be -x i.e. a negative number, when we have just learnt that |x| is always positive.

Please note that |x| is equal to -x only when x is negative and in such a case -x would

eventually turn to be positive, e.g. when we have to find |-4|, here x = -4, a negative number.

Using |x| = -x |-4| = -(-4) = 4

Solving Equations involving Mods

Whenever an equation involves mods, the theoretical way is to consider two cases viz. once the quantity, of which the mod is taken, is considered positive and once it is considered negative. Then using the definition of mod, the sign of mod is eliminated and then the equation is solved as usual.

Be very careful of checking if the solution arrived at satisfies the condition of the quantity within the mod being positive or negative, as the case may be. An example will make the above working much more clearer.



E.g. 34: Solve for 2x + |x| = 6



Thus solution to 2x + |x| = 6 is x = 2.

Obviously, we would not be able to write the solution in as much details as above, but the process is always the same.

Further do not get wrong ideas that only one of the branches would yield a solution, it is quite possible that solutions to both branches satisfy the conditions and thus there could be more than one solution and it is equally possible that neither of them yield a solution in which case there exists no real value of *x* that satisfies the equation.

E.g. 35: Solve for x; $x^2 + 5|x| + 6 = 0$



Thus there is no solution to $x^2 + 5|x| + 6 = 0$



And obviously there would be no solution because x^2 as well as |x| is positive, so how

can two positive quantities when added to 6 become zero?

However if the question was...

E.g. 36: Solve for x; $x^2 - 5|x| + 6 = 0$



Thus the solutions to $x^2 + 5|x| + 6 = 0$ are 2, -2, 3, -3

E.g. 37: Solve for |x-3| = 2

In this case, the conditions to be assumed are not if x is positive or negative but if (x - 3) is positive or negative, because we are taking the mod of (x - 3) and not of x.

As studied in inequalities, the condition x - 3 > 0 i.e. x > 3 and x - 3 < 0 i.e. x < 3. One should be very adept at simplifying such simple inequalities in the mind itself and should not waste time writing these conditions.

If x > 3, then x - 3 = 2 i.e. x = 5. Since x = 5 satisfies the condition x > 3, it is a solution

If x < 3, then -(x - 3) = 2 i.e. -x + 3 = 2 i.e. x = 1. Since x = 1 satisfies the condition x < 3, it is also a solution.

Thus solution to |x-3| = 2 is x = 1 or 5

The text below explains a very elegant method to arrive at this solution though the use of logic. So make sure you understand it well, it will be used very often here-after.



Understanding Modulus as 'Distance'

|x| could also be understood as the distance of the point x from 0, zero, on a number line.

|4.5| = 4.5 could be visualized as the distance between 0 and 4.5 as follows:



By the same logic we can understand |x - a| as the distance of the point *x* from *a* |5 - 1.5| = 3.5



Extending this logically, |x + a| would imply the distance of x from -a|1 + 4| = 5



Considering $|x \pm a|$ as distance of x from $\mp a$ is a very useful technique. Consider the last example that we solved, |x-3| = 2

By this equation we are finding all those x such that distance of x from 3 is equal to 2. Now distance could be either on the right hand side or on the left hand side. Thus the solution should be obvious if you visualize the following





The answer has to be 1 and 5.

E.g. 38: Solve for *x*; |x+5| = 8

We need x such that distance from -5 is 8. From -5, a distance of 8 on the left will give us -13 and from -5, a distance of 8 on the right will give us 3. Thus the answer is -13 and 3.

E.g. 39: Solve for *x*; |2x-5| = 3

Just because there is a 2x instead of x should not change the logic. Considering 2x as a point, we get 2x is/are those points whose distance from 5 is 3. Thus 2x = 2 or 2x = 8 $\Rightarrow x = 1$ or x = 4.

E.g. 40: Solve for *x*; |3x+1| = 5

A distance of 5 from the point -1 will give us -6 and 4. Thus x = -2 or $\frac{4}{3}$

Read through the above examples again, but make sure that you can solve questions just as done in the last solved example.

Mods with Inequalities

E.g. 41: Solve for *x*; $|x-4| \le 3$

The logical meaning of this questions is to find all the points i.e. x, whose distance on the number line from the point 4 is less than or equal to 3.

We have learnt that 1 (i.e. 4-3) and 7 (i.e. 4+3) are two points on the number line which are at a distance of 3 from the point 4. Since the distance could also be less than 3, all points that lie between 1 and 7 are also acceptable. Thus the solution to this equation is $1 \le x \le 7$. The above can also be understood from the visual aid given below...


E.g. 42: Solve for *x*; |2x-1| > 7

In this case '2x' can be any point which on the number line lies at a distance greater than 7 from the point 1. Thus, 2x can be any point to the left of (1 - 7) i.e. to the left of -6 or 2x could be any point to the right of (1 + 7) i.e. 8. Check if you can visualize that when distances are measured from the point 1, all points to the left of -6 and to the right of 8 are at a distance greater than 7.

Thus 2x < -6 or 2x > 8 i.e. x < -3 or x > 4.

The above can easily be remembered and generalized by the following step-wise:

|x|: the distance of x from 0.

 $|x| = a \Longrightarrow x = -a$ or a, distance of x from 0 is a



 $|x| < a \Rightarrow -a < x < a$, distance of x from 0 is less than a, so all points between -a and a



 $|x| > a \Rightarrow x < -a$ or x > a, distance of x from 0 is more than a, so all points less than -a or all points more than a



|x-a|: the distance of x from a

 $|x-a| = b \Rightarrow x = a - b \text{ or } a + b$, distance of x from a is b





 $|x-a| < b \Rightarrow a-b < x < a+b$, all point within a distance of *b* from *a*.



 $|x-a| > b \Rightarrow x < a-b \text{ or } x > a+b$, all point less than a-b or all point greater than a+b.



Equations involving two mods

E.g. 43: Solve for *x*; |x-3| + |x+2| = 4

We shall first solve these questions using the theoretical approach and then learn the logical way, if it can be applied. Knowing the theoretical process is important as in many situations the logical process won't work, so make sure you do not discount the theoretical approach. It may be the only way out in many situations.

In questions which involve more than two mods, we would have to consider the case when each of quantity, for which mod is to be found, as positive and negative. Thus, if there are two mods, theoretically we would have a total of four conditions, as shown below:

	<u>x-3</u>	+ x+2		<u>x-3</u>	+ <u>x+2</u>
Case i	+ve	+ve	Case iii	-ve	+ve
Case ii	+v8	- <i>v</i> e	Case iv	-ve	- <i>v</i> e

However, practically one of the four cases is impossibility. Before reading ahead, think which one of the four cases is not possible?

For x - 3 would become zero when x = 3. For all x < 3, it will be negative and for all x > 3, it will be positive.

Similarly x + 2 will become zero when x = -2. For all x < -2, it will be negative and for all x > -2, it will be positive.

If your answer was case ii as the case which is not possible, you are right. For x - 3 to be positive, x has to be greater than 3 and if x > 3, then x + 2 cannot be negative?



How can one uncover such situations that are not possible? A simple way in which you would not miss any such insights is to follow the approach given below, instead of considering all possible combination of each quantity in mod to be positive and negative.

On a number line plot the points where each of the quantity within the Mods become equal to zero. In our cases, the points are -2 and 3.



Now the entire range of x is divided into three regions. In each of these three regions, check the polarity of each of the term i.e. x - 3 and x + 2, in our example. Whatever combination of polarities you get are the only feasible ones.



Now, once you know the polarity of (x-3) and (x+2), you can easily eliminate the modulus sign by using the formula of mod and then solve the equations. Please do not forget to check if the solution so obtained satisfies the condition you started with.

E.g. 44: Solve for |x-3| + |x+2| = 4



x < -2, so not a solution

x > 3, so not a solution

Thus there is no solution to |x-3| + |x+2| = 4



The above method is the theoretically right method.

However if all the sign between the mods used is +, plus, sign, there is a shorter, elegant and logical solution as well.

Remember that |x-3| refers to the distance of x from 3 and |x+2| refers to the distance of x

from -2.

Thus, if x > 3, then



Irrespective of how close or how far x lies from 3, if x is to the right of 3, the two distances have to add up to more than 5 and thus |x-3|+|x+2| can never be equal to 4

If -2 < x < 3, then



Again irrespective of where x lies, as long as it lies within -2 and 3, the two distances will always add up to 5 and thus |x-3|+|x+2| can never be equal to 4.

If x < -2, then



Irrespective of how close or how far x lies from -2, if x is to the left of -2, the two distances have to add up to more than 5 and thus |x-3|+|x+2| can never be equal to 4

Thus we have seen that whichever value x assumes, |x-3| + |x+2| can never be equal to 4. Thus we have no solution for the equation |x-3| + |x+2| = 4.



Let's say the equation was |x-3| + |x+2| = 7

If x > 3, then



and the two distances marked by the arrows have to add up to 7. It's obvious that the distance between -2 and 3 is 5 (the dashed line in the below figure) and thus,



the two arrows should now add up to 2. And since they are equal in lengths, each will be 1 unit and thus *x* has to be 4. You can check by substituting x = 4 in the equation if you are not convinced that this is a solution to |x-3|+|x+2|=7

If -2 < x < 3, then



Again irrespective of where x lies, as long as it lies within -2 and 3, the two distances will always add up to 5 and thus |x-3|+|x+2| can never be equal to 7.

If x < -2, then



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the two distances marked by the arrows have to add up to 7. Again using the logic that distance between -2 and 3 is 5 (the dashed line in the below figure) and thus,



the two arrows should now add up to 2. And since they are equal in lengths, each will be 1 unit and thus *x* has to be -3. You can check by substituting x = -3 in the equation.

Thus the solution to |x-3| + |x+2| = 7 would be x = -3 or 4.

The above approach could also be used to solve inequalities...

Had the above question been |x-3|+|x+2| < 7, the only difference is that now the sum of the distances has to be less than 7. It should not be too difficult a task to realize that the solution would be -3 < x < 4.

Alternately if the question had been, |x-3| + |x+2| > 7, the sum of the distances has to be more than 7 and thus the solution would be x < -3 or x > 4.

The above method can also be used to solve equalities or inequalities involving more than 2 Mods. The only condition is that the sign between all mods should be that of +, plus. To understand the solution for equation or inequality and to make it more easy, it would be worthwhile to extend the above logical solution with an example...

Say three friends A, B and C are staying at three points A, B and C, in a straight line with B between A and C. Further let the distance between A and B be 4 units and the distance between B and C be 6 units.



Answer the following questions based on the above conditions. In the following questions the total distance refers to the sum of the distances that *A*, *B* and *C* travel.

1. If they have to meet at any point between *A* and *C*, what is the total distance traveled by *A* and *C* together?

2. What is the minimum total distance that they have to travel so that all three meet at the same point? At what point would they meet in this case?



3. If they meet at *C*, what is the total distance that they have to travel?

4. If they meet at *A*, what is the total distance that they travel?

5. If they have to meet after traveling a total distance of 20 units, at what point should they meet?

6. If they have to meet at a point 1 km away from *B*'s place, towards *A*, what is the total distance that they travel?

7. If they have to meet after traveling a total distance of 12 units, where should they meet?

8. Where should the meeting point be if the total distance to be traveled has to be 8 units? Solutions:

1. As seen earlier, if there are only two points and x, the meeting point in this case, is between the two points, the sum of the two distance will always be a constant equal to the distance between the two points. To refresh, see the below visual.



in any case the sum of the distance traveled by *A* and *C* is 10 units i.e. if they meet between *A* and *C*.

2. To travel the minimum distance, it is obvious that they would meet somewhere between A and C (and not to the left of A or to the right of C). And as seen in the above case, if they meet at a point somewhere between A and C, the total distance traveled by A and C will always be 10 irrespective of the meeting point. Thus, so that the distance traveled by all of them be least, why make B travel at all? So they will meet at B's place and the minimum distance they have to travel will be 10.

3. A will travel 10 and B will travel 6 and C will not travel at all. So total distance is 16

4. *C* will travel 10 and *B* will travel 4 and *A* will not travel at all. So total distance is 14

5. To travel a total distance of 20, it should be obvious that they have to meet to the left of A or to the right of C. Obvious because in the earlier two questions we calculated that if they meet at A, the total distance traveled is only 14 units. So to travel a total of 20 units, they have to cover a distance of 6 more units. But now all three will travel the distance to the left of A. Thus each of



them should travel 2 units more to the left of A. And thus they should meet at 2 units to the left of A.

Similarly, when all three meet at C, they have already covered 16 units. To cover further 4 units, all three have to travel beyond C, i.e. to the right of C and together have to cover 4 units. Thus,

the meeting point has to be $\overline{3}$ units to the right of C.

4

So the answer is either at 2 units to the left of A or 3 units to the right of C

6. *A* will travel 3 units, *B* will travel 1 unit and *C* will travel 7 units. So the total distance covered is 11 units.

7. The distance to be covered is 12 in this question. It should be realized that this is more than the minimum distance to be covered and thus the meeting point will be other than *B*. At the same time the distance to be covered is less than 14 units (total distance to meet at *A*) or 16 units (total distance to meet at *C*). Thus the meeting point when they travel a distance of 12 has to be somewhere between *A* and *C*.

Whenever the meeting point is between the two end-points, it is a very tricky situation. The approach to be followed always in such situations is to consider a pair of persons, one each from opposite ends. The pair would together cover a distance equal to the distance between the end-points, irrespective of which point they meet, as long as they are meeting in-between.

If the meeting point is somewhere between A and C, then A and C would travel 10 units, irrespective of where the meeting would be. Thus the 2 units extra has to be traveled by B and the meeting point would be 2 units to the left or to the right of B.

8. This question is just a reminder that the minimum distance to be covered for all three of them to meet is 10 units. So there is no meeting point where all three meet after traveling a distance of 8 units.

Having understood the above analogy, now you should understand how each of the above questions can be asked using Mod notation.

The total distance traveled by *A*, *B* and *C* with the scenario of the distance at which they are staying can be captured in |x+3|+|x-1|+|x-7|. This is because distance between -3 and 1 is

4 units and distance between 1 and 7 is 6 units. Thus we can consider that A stays at point -3, B stays at point 1 and C stays at point 7.

Now for the equivalent questions are as follows. Not all the above questions are translated but the important ones are done. In the above, since it was a learning experience, there were many more leading questions like questions, 1, 3, 4 and 6.



- 2. Find the minimum value of |x+3| + |x-1| + |x-7|. For what value of x does this minimum occur?
- 5. Solve for x; |x+3| + |x-1| + |x-7| = 20
- 7. Solve for x; |x+3| + |x-1| + |x-7| = 12
- 8. Solve for x; |x+3| + |x-1| + |x-7| = 8

Solve these theoretically and then compare the two approaches for a more holistic learning. The same approach can also be used even if the coefficient of x is something other than 1... **E.g. 45:** Find the minimum value of |2x-3|+|x+5|

The expression |2x-3|+|x+5| can be thought as $2 \times \left|x-\frac{3}{2}\right|+|x+5|$ and thus in terms of our understanding of Mods as distances it implies that the distance from $\frac{3}{2}$ has to be taken twice.

Translating it to our analogy of distances to be traveled by friends to meet, it means two friends are staying at the point $\frac{3}{2}$



Thus, for the minimum distance to be traveled, the best option is that the one friend staying at -5, A, walks the entire distance to meet the two friends, B & C, staying at $\frac{3}{2}$. If A does not walk the entire distance, whatever distance he stops short of has to be traveled by both B and C and this would add twice the distance to the total distance traveled, instead of being added just once had A himself traveled that distance.

Thus the minimum distance to be traveled is the distance between -5 and $\frac{3}{2}$ i.e. $5 + \frac{3}{2} = 6.5$

and this minimum value would be when they meet at B or C's place i.e. at $\frac{5}{2}$.

In terms of the questions, this means the minimum value of |2x-3| + |x+5| will be 6.5 and will 3

occur when $x = \frac{3}{2}$

(you should take other values of x and check this up for a more thorough understanding)



E.g. 46: Solve for *x*; |3x-6| + |2x+5| = 15

Using our analogy, this means three friends are staying at 2 and two friends are staying at -2.5.



The distance between them is 4.5 units.

Searching for a solution on the left hand side, if they meet at *A*'s (or *B*'s place, both are same), the total distance traveled is 13.5. To travel a distance of 15, they have to cover 1.5 more and hence they will meet to the left of *A*'s place. Going leftwards from *A*'s, all five of them will travel and hence each will travel $\frac{1.5}{5} = 0.3$. Thus when they meet at -2.5 - 0.3 = -2.8, the total distance covered will be 15.

Searching for a solution on the right hand side, if they meet at *C*'s place, the distance already covered will be $2 \times 4.5 = 9$. They yet have to cover 6 and all the five of them will together cover this extra distance, going rightwards of *C*'s place. Thus each will cover $\frac{6}{5} = 1.2$. Thus when they meet at 2 + 1.2 i.e. 3.2, they would together have covered a distance of 15.

Thus solution to |3x-6| + |2x+5| = 15 is x = -2.8 or 3.2

E.g. 47: Solve for *x*; |2x+6| + |3x-6| < 12



The distance between the two points is 5. The minimum distance to be traveled to meet is when *A* and *B* go to the other end and this minimum distance is $2 \times 5 = 10$. Since the question is asking that the distance has to be less than 12, it would surely have a solution.

Let's say *A* and *B* travel to the other end. A distance of 10 has already been covered. They, all five of them, could go a little more further to the right because they can cover a distance upto 12. Thus each of them could go further rightwards by $\frac{2}{5} = 0.4$. Thus on the right end, they can go upto 2.4 (2.4 is not inclusive as the distance to be traveled has to be strictly less than 12)



Towards the left, if C, D and E travel to the other end, they would travel 15 and hence the leftmost that they can go, traveling less than 12, would lie somewhere in between the two points. Whenever the solution is between the points, it is a tricky situation and as already learnt, the approach is to consider pairs of friends.

A and C together will cover a distance of 5 at whichever point, in between the two end-points, they meet. Similarly B and D would also cover 5. Thus the total distance traveled by A, B, C and D is 10. The remaining distance of 2 has to be covered by E and thus the left most point they could meet will be 2 units to the left of 2 i.e. 0 (again 0 not inclusive because of the strict inequality)

Thus the solution to meet after traveling a distance of 12 units is to meet between points 0 and 2.4

And so the solution to |2x+6| + |3x-6| < 12 is 0 < x < 2.4

Limitation of the above approach

Unfortunately the above logical process cannot be used when there is a – (minus) sign between any Mod e.g. |x+3| - |x-1| + |x-7| = 8 has to be solved theoretically or graphically, as we shall next.



Exercise

1.	How many integral values of <i>x</i> satisfy the relation $ 2x-7 < 5$								
	(1) 4	(2) 5	(3) 6	(4) 7					
2.	How many integral values of x satisfy the relation $ x^2 - 5x < 6$								
	(1) 8	(2) 4	(3) 2	(4) 5					
3.	Find the minimum value of $ x-2 + x-5 + x+2 $								
	(1) 11	(2) 7	(3) 10	(4) 0					
4.	For which of the follow least value?	ying value of x , does the	expression $ x-2 + x +$	x+1 + x+3 assume the					
	I. $x = 0$	II. $x = -1$	III. $x = -0.5$	IV. $x = 0.5$					
	(1) III	(2) both I and II	(3) I, II and III	(4) All four					
5.	5. Solve for $x: 2 - x + x + 5 = 7$								
	(1) -2 < x < 5	$(2) -2 \le x \le 5$	(3) - 5 < x < 2	$(4) -5 \le x \le 2$					
6.	For how many of the following value of k does the equation $ x - 2 + x - 5 + x + 4 = k$ have exactly two integral solutions.								
	I. <i>k</i> = 9	II. $k = 2$	III. $k = 10$						
	IV. <i>k</i> = 16								
	(1) 0	(2) 1	(3) 2	(4) 3					
7.	For what value of x does the expression $ 3x - 6 + 2x + 4 $ assume the least value?								
	(1) -2	(2) 2	(3) 2 or –2	$(4) -2 \le x \le 2$					
8.	Find the least value of	x-1 + x-2 + x-3 +	$ x-4 + \dots + x-99 $						
	(1) 0	(2) 50	(3) 1225	(4) 2450					
9.	For what value of k wo	uld the equation, $ 2x - 1 $	+ x+2 = k, have exact	tly one solution					
	(1) 0.5	(2) 1	(3) 2	(4) 2.5					



Graphs of Expression involving Mods

Consider the graph of y = |x|.

By definition we have y = x, if $x \ge 0$ and y = -x, if x < 0

Plotting the line y = x and y = -x,



However y = |x| is y = x only when $x \ge 0$. Thus the graph of y = |x| would not be the entire line of y = x but only that part of the line which lies in the region $x \ge 0$.

Similarly the graph of y = |x| would only be that part of the line y = -x which lies in the region where x < 0.

Thus the graph of y = |x| would be only the bold line in the following figure...



From now on the graph of |x| would be shown simply as a V.





Graph of |x| + |y| = a

The expression |x| + |y| = a is equivalent to the following four relations, along with the conditions on x and y...

x + y = a if $x \ge 0$ and $y \ge 0$, i.e. first quadrant x - y = a if $x \ge 0$ and y < 0, i.e. fourth quadrant -x + y = a if x < 0 and $y \ge 0$, i.e. second quadrant -x - y = a if x < 0 and y < 0, i.e. third quadrant

The four lines are plotted as dotted lines in the following graph. But only that part of the line is going to be the graph of |x| + |y| = a, which satisfies the above conditions. The graph of |x| + |y| = a is shown by the bold line.



Thus the graph of |x| + |y| = a will be a square (rotated at 45 degrees) with diagonal lengths 2a i.e. the side will be $\sqrt{2}a$

To get an idea of what sort of questions could be asked with graphs, consider this CAT question...

E.g. 48: What is the area enclosed by the graph of |x| + |y| = 4

Based on the above discussion, the area would be $(4\sqrt{2})^2 = 32$

E.g. 49: Find the number of integer solutions to |x| + |y| = 5

For such types of questions the easiest approach is to count and try to see if we get any pattern.

Obviously x cannot be 6 or greater or else |y| would have to be negative.

Thus if $x = \pm 5$, y could only be zero i.e. 2 solutions

If $x = \pm 4$, y could be ± 1 i.e. 4 solutionsIf $x = \pm 3$, y could be ± 2 i.e. 4 solutionsIf $x = \pm 2$, y could be ± 3 i.e. 4 solutionsIf $x = \pm 1$, y could be ± 4 i.e. 4 solutionsIf x = 0, y could be ± 5 i.e. 2 solutions

Thus there are a total of 20 solutions.



These solutions refer to those points on the graph of |x| + |y| = 5 which have integral solutions...



While the graph does not make our job of finding the solutions any easier, understanding the meaning of the 20 solutions just found is helpful in understanding other similar situations.

Shifts in Graph positions

We know the graph of y = |x| is a V shaped graph.

How would the graph of y = |x| + a look like?

It should not be very difficult to imagine that each earlier value of y (that of y = |x|) would get incremented by a. And thus, the entire graph would shift upwards by a (assuming a to be positive)

Thus the graph of y = |x| + 4 would look like..



0, -4

and the graph of y = |x| - 4 would look like...



E.g. 50: What would be the area of the region enclosed by the graph of y = |x| - 4 and the X-axis?

As seen in above figure, the region would be a triangle of height 4. To find the area we also need to know the base.

To find the co-ordinates where the graph cuts the *X*-axis, put y = 0 in its equation. Since y = |x| - 4, putting y = 0 will give us |x| = 4 i.e. x = 4 or -4.

Thus the base would be of length 8 and the required area would be $\frac{1}{2} \times 8 \times 4 = 16$

Even if you have understood the above method of finding the co-ordinates of the points where the graph cuts the *X*-axis (by putting y = 0) or the *Y*-axis (by putting x = 0), it would still be advisable to also get to know the approach of finding the co-ordinates using the funda of slope. This approach is more logical and does not entail the use of pencil...

Since the two inclined lines have a slope of 1, when *y* changes from -4 to 0 (an increase of 4), for the positively sloped line, *x* would also increase by 4 i.e. *x* would change from 0 to 4; and for the negatively sloped line, *x* would decrease by 4 i.e. *x* would change from 0 to -4. This approach is explained in details, i.e. for other values of slope as well, after the next graph.

If the above funda of slope is not clear, you could also use the fact that the lines are inclined at 45 degrees to the horizontal (or the vertical) and use the property of a 45 - 45 - 90 triangle. But this aspect of 45 degrees can be used only when slope is 1. When slope is 2, you would have to use the funda of change in *y*-coordinates and corresponding change in *x* coordinates.

What would be the graph of y = |x - 4|?

When x = 4, y = 0. When x = 3 or 5, y = 1 and so on. Thus we can see that this is a horizontal shift and the V shaped graph will shift such that the 'vertex' of the V will now be at 4.

Combining the two shifts, vertical and horizontal, it should not be too difficult to immediately draw the graph of y = |x - 3| + 5 as follows:





If the above is not understood first draw the graph of y = |x - 3|, i.e. horizontally shift the V such that vertex is at 3. Now consider y = |x - 3| + 5 i.e. shift the graph vertically by 5.

The above can also be obtained by considering the equation y = |x - 3| + 5 as two equations i.e.

y = (x - 3) + 5, if $x \ge 3$ and

y = -(x - 3) + 5, if x < 3.

Concept of slope and its use in finding the coordinates of points

While discussing the basics of algebra, we briefly learnt the concept of slope being the inclination of the line. To formally define the slope...

Slope of a line is the change in y co-ordinate for a unit change in x co-ordinate. i.e. if x increases by 1 units, 'by how many units does y increase?' would answer the slope of the line.

Thus, slope = $\frac{\text{Change in } y \text{ co-ordinate}}{\text{Change in } x \text{ co-ordinate}}$

If for one unit of change in x, y changes by a small amount, the slope would also be a small value and the line would be flatter. However when x changes by 1 units and this causes a large change in y, the line would be quite inclined and the slope would be a higher value...



Further slope could be positive or negative....

When a line is inclined upwards, while going left to right, it is said to have a positive slope. Slope is positive in this case because an increase in x causes an increase in y and a decrease in x causes a decrease in y.

And when a line is inclined downwards, while going left to right, it has a negative slope. Slope is negative in this case because an increase in x causes a decrease in y and a decrease in x causes an increase in y.





The slope of the *X*-axis (or any horizontal line) is 0, zero; and that of the *Y*-axis (or any vertical line) is ∞ (infinite)

Lastly, the following figure gives us the idea of the magnitude of the slope for various lines...



We started learning the concept of slope to find the co-ordinates of a point when the slope and any other coordinate are known...





Since the slope is 3, you should think in your minds that $\frac{\text{Change in } y}{\text{Change in } x} = 3$. Using this and

knowing any one of change in x or in y, the change in the other can be found out. Like in the above example it can be known that change in x is $\frac{5}{3}$ and thus the x coordinate to be found will

be
$$0 + \frac{5}{3} = \frac{5}{3}$$
.

Another example...

In the following figure, what is the coordinates of point B, if the slope of the line is

$$-\frac{2}{3}$$
 and co-ordinates of A is (-8, 0)



Slope =
$$\frac{\Delta y}{\Delta x} \Rightarrow -\frac{2}{3} = \frac{?}{8}$$

Thus, change in y will be $-\frac{2}{3} \times 8 = -\frac{16}{3}$. Thus co-ordinate of B is $\left(0, -\frac{16}{3}\right)$

E.g. 51: What is area enclosed by the graph of y = |2x + 6| - 8

The graph of y = |2x + 6| - 8 should immediately be identified as...



Since the slope is 2, for a change of 8 in *y*, *x* has to change by 4 and thus the a = -3 + 4= 1 and b = -3 - 4 = -7.

Thus the base of the triangle is the distance between -7 and 1 i.e. 8. And area = $\frac{1}{2} \times 8 \times 8 = 32$



Graphs of expression involving more than one Mods with minus sign between them

To wind up the topic, let's consider the one case we have left so far...

E.g. 52: Solve for x; |x + 4| - |x| - |x - 3| = 12.

As noted earlier, since there is a minus sign between the Mods, we cannot use our analogy of friends meeting after traveling a distance of 12. Thus the only way to solve the question is the theoretical approach by using the definition of |x|. Just to recapture...



Thus there is no real solution to |x + 4| - |x| - |x - 3| = 12

Obviously it takes a hell lot of time to do the above process. A marginally better process is to get an idea of the graph of |x + 4| - |x| - |x - 3|. At least this will tell us in which region would the solution lie.

To plot the graph, one has to approach it almost similarly like the above, but it is much faster if one remembers the following...

The line ax + b is...

A upward inclined line (going left to right) if *a* is positive and a downward sloping line if *a* is negative.

a would denote the slope of the line. If magnitude of *a* is 1, the line will be at 45 degrees to the horizontal and if magnitude is less than 1, it will be flatter and if magnitude is greater than 1, the line will be a steeper line.



To plot the graph quickly, first plot the points for each value of x where each of the quantity in mod becomes zero i.e. for x = -4, 0 and 3 in this case...



• (-4, -11)

The *Y*-axis is not drawn, but you can easily get the relative placement as the co-ordinates are written.

Now obviously in each region the expression is going to be a straight line. Thus for the regions that lies in-between the points, we just have to join the points.

And for the two regions on either side, you can work in the following way...

When x > 3, the expression would simplify to something of the sort ax + b. To save time just find what would be the coefficient of x. Do not waste yr time in finding the entire expression ax + b. In our example it would be -1 i.e. a decreasing line with slope -1.

Alternately find the value of the expression for any *x* greater than 3, say for x = 4, the expression boils to 3. Thus plotting the point (4, 3) we get a downward sloping line of slope -1.

When x < -4, the expression would again simplifies to ax + b and in our example, the coefficient of *x* would be 1 i.e. a increasing line with slope 1.

Alternately find the value of the expression for any *x* less than -4, say for x = -5, the expression boils to -12. Thus plotting the point (-5, -12) we get a line going downward as we travel leftwards i.e. a increasing line (from left to right)

Thus the graph of |x + 4| - |x| - |x - 3| would look like...







Thus we can just see the graph and know for sure that for no value of x would the y co-ordinate be equal to 12.

The graph also helps us in getting a complete picture. Thus if the question had been what is the maximum value of |x + 4| - |x| - |x - 3|, the graph would have made it very apparent that the maximum value would be 4 and it would occur when x = 3.

Using the graph you could also find for what particular value of x does the expression take a desired value. Let's see this in a last example...

E.g. 53: Solve for x; |2x + 6| - |x - 2| = 12

Let's plot the graph of |2x + 6| - |x - 2|

Finding the two points when 2x + 6 = 0 and x - 2 = 0...

When x = -3, the expression is -5 and when x = 2, the expression is 10

When x < -3, the expression will be a line with slope -1 and when x > 2, the expression will be a line with slope 1

Thus, the graph would look like



To find for what value of x would the expression be equal to 12...

To find the solution in the left-most region, start with the point (-3, -5). From this point the *y* co-ordinate has to increase to 12 i.e. an increase of 17. Since the line has a slope of -1, *x* should correspondingly decrease by 17 and thus the *x* co-ordinate of the point where *y* co-ordinate will be 12 is -3 - 17 = -20.

Check: Substituting x = -20 in |2x + 6| - |x - 2|, we get |-34| - |-22| = 34 - 22 = 12.

To find the solution in the right-most region, start with the point (2, 10). From this point the *y* co-ordinate has to increase to 12 i.e. an increase of 12. Since the line has a slope of 1, *x* should also correspondingly increase by 2 and thus the *x* co-ordinate of the point where *y* co-ordinate will be 12 is 2 + 2 = 4.

Check: Substituting x = 4 in |2x + 6| - |x - 2|, we get 14 - 2 = 12.

Thus the solution for |2x + 6| - |x - 2| = 12 is x = -20 or 4



Exercise

10.	Find the difference bet	ween the maximum and	minimum value of $ x - 2 $	- x+1	
	(1) 1	(2) 2	(3) 3	(4) 6	
11.	Find the number of inte	eger solutions to $ x - 1 $ +	y+1 < 5		
	(1) 61	(2) 41	(3) 40	(4) 50	
12.	Find the area enclosed	by the graph of $y = 8 - $	x-5 and the X-axis		
	(1) 64	(2) 40	(3) 32	(4) 25	
13.	Find the area enclosed	by the graph of $y = \left \frac{x}{2} \right $	3 - 5 and the X-axis		
	(1) 12.5	(2) 25	(3) 50	(4) 75	
14. Find the number of integer solutions that satisfy $y - x - 3 > -5$ and $y < 0$					
	(1) 25	(2) 16	(3) 14	(4) 9	
15.	Find the area enclosed	by the graphs of $ y = x $	-3 and x = 5.		
	(1) 2	(2) 4	(3) 8	(4) 16	



Maxima Minima

Broadly there are four types of questions in which the maxima or minima of a function has to be found...

1. Maxima or minima of a polynomial function, usually only upto the third degree

2. Maximum of the minimum of two or more expressions - you first find the minimum value of two or more expressions and then find the maximum among these minimum values found out. The question could be any thing like maximum of minimum values, or minimum of maximum values or maximum of maximum values, or minimum of maximum values. For us to refer to such questions we will call the Max-Min type of questions.

3. The maximum or minimum of a sum or product of positive variables. These questions uses the concept of Arithmetic Mean of a set of positive numbers is always greater than or equal to the Geometric Mean of the set of numbers. Thus, to represent such questions we shall use the term – Maximum and Minimum based on AM, GM

4. The last type of questions are those where we have to find minima of expressions involving mods. This topic has been covered in details on the topic of mods and so will not be covered here.

Maxima or Minima of a Polynomial Function

We have already learnt that about graphs of polynomial functions and have studied in details, the graphs of a quadratic and a cubic expression. The point where the graph of a polynomial function changes its direction, from moving upwards to moving downwards, is called a maxima. Similarly the point where the graph of a polynomial function changes its direction, from moving downwards to moving upwards, is called a minima. The maxima and minima for a quadratic function and a cubic function are shown in the figure below:

Having studied quadratic polynomials in depth, it should not be too surprising to note that for a quadratic function, $ax^2 + bx + c$, when *a* is positive, it would have a minima and when *a* is negative it would have a maxima.

A cubic function could have both a maxima and a minima or could have none (remember the graph of x^3 , where the troughs and crests are smoothed out).







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Maxima/Minima of a Quadratic Expression:

For either of the two cases (the quadratic having a maxima or a minima), the maxima or the minima, as the case may be, will occur when $x = -\frac{b}{2a}$

E.g. 54: Would the expression $18x - 3x^2 + 8$ have a maxima or a minima? For what value of x would the maxima or minima occur? What is the maxima or minima value?

Since *a*, the coefficient of x^2 is negative, the expression would have a maxima value.

The maxima value would occur when $x = -\frac{b}{2a} = -\frac{18}{2 \times (-3)} = 3$

To find the maxima value, substituting x = 3 in the expression we get $18 \times 3 - 3 \times 3^2 + 8 = 54 - 27 + 8 = 35$.

E.g. 55: For what value of *a* would the expression, $ax^2 - 8x + 4$, have the minimum value equal to zero?

Since the quadratic expression has a minimum value, *a* has to be positive.

Further a quadratic expression would assume a minimum value of 0 only if its determinant is zero, thus $64 - 16a = 0 \implies a = 4$.

Alternately if you could not think of the above, the expression would assume a

minimum value when $x = -\frac{-8}{2a} = \frac{4}{a}$. Substituting this in the expression and knowing

that the minimum value is zero, $a \times \frac{16}{a^2} - 8 \times \frac{4}{a} + 4 = 0 \implies 16 - 32 + 4a = 0 \implies a = 4$

The derivative way to find the maxima or minima:

Please note that you need not know the derivative method to find the maxima or minima of a quadratic expression, as learnt above. And in the entrance test also, there has never been a question so far that needed you to use derivatives to find the maxima or minima value. Thus you could safely ignore this part of the chapter. It is given here just to satisfy the need of those who immediately associate maxima and minima with Derivatives. But a word of caution to them as well: You would be wasting your preparation time and also wasting time in the exam if you use the derivative approach.

Consider any polynomial function in variable *x* and denoted by *y*. If you want to find the maxima or minima of this polynomial function, the standard method using derivatives is explained below.



Step 1: Find the first derivative of y with respect to x, i.e. y'

Step 2: Solve for y' = 0 and let's say the roots are $x = x_1$ or x_2 or x_3 ...

Step 3: Find the second derivative of y with respect to x, i.e. y''

Step 4: Taking *x* equal to each of the value of roots found in step 2, turn by turn, find the value of y''. With *x* being equal to a root, if y'' > 0, the polynomial will assume a minima value for *x* being equal to that particular root; if y'' < 0, the polynomial will assume a maxima value for *x* being equal to that particular root.

You would need to refresh the following derivative functions:

1. If $y = x^n$, then $y' = n x^{n-1}$. If $y = ax^n$, then $y' = an x^{n-1}$

2. If
$$y = ax^n + bx^{n-1} + cx^{n-2} + \dots$$
, then $y' = anx^{n-1} + b(n-1)x^{n-2} + c(n-2)x^{n-3} + \dots$

E.g. 56: Find the maxima and/or minima value of $4x^3 + 5x^2 - 2x + 6$.

$$y = 4x^{3} + 5x^{2} - 2x + 6$$

$$y' = 12x^{2} + 10x - 2$$

$$y' = 0 \text{ i.e. } 12x^{2} + 10x - 2 = 0$$

i.e. $6x^{2} + 5x - 1 = 0 \Rightarrow 6 \times (x + 1) \times (x - \frac{1}{6}) = 0 \Rightarrow x = -1 \text{ or } \frac{1}{6}$
Next, $y'' = 24x + 10$
For $x = -1$, $y'' = -24 + 10 = -14$ i.e. $y'' < 0$
Hence at $x = -1$, y would assume a maximum value.
Substituting $x = -1$, the maximum value of $4x^{3} + 5x^{2} - 2x + 6$ is $-4 + 5 + 2 + 6 = 9$
For $x = \frac{1}{6}$, $y'' = 4 + 10 = 14$ i.e. $y'' > 0$
Hence at $x = \frac{1}{6}$, y would assume a minimum value.
Substituting $x = \frac{1}{6}$, the minimum value of $4x^{3} + 5x^{2} - 2x + 6$ is
 $\frac{1}{54} + \frac{5}{36} - \frac{1}{3} + 6 = \frac{2 + 15 - 36 + 648}{108} = \frac{629}{108}$.



It is likely to be argued that the only way to find the maxima or the minima value of a cubic polynomial function is by using the derivative approach. While this is generally true, in the rare case where one needs to find the maxima or a minima of a cubic expression in entrance exams, in those specific cases there is a better alternative available. This is based on the concept of Arithmetic and Geometric Mean and will be discussed while in the relevant part there.

A unique question asked in entrance exam is:

E.g. 57: Find the maximum and/or minimum value of $\frac{x^2 - x + 1}{x^2 + x + 1}$ for real values of x.

And the standard way to solve this is...

Let
$$\frac{x^2 - x + 1}{x^2 + x + 1} = k \implies x^2 - x + 1 = kx^2 + kx + k$$

 $\Rightarrow (1 - k)x^2 - (1 + k)x + (1 - k) = 0$

Now for *x* to be real, the determinant has to be greater than or equal to zero...

$$(1+k)^{2} - 4(1-k)(1-k) \ge 0$$

$$\Rightarrow (1+2k+k^{2}) - 4(1-2k+k^{2}) \ge 0 \quad \Rightarrow -3k^{2} + 10k - 3 \ge 0$$

$$\Rightarrow 3k^{2} - 10k + 3 \le 0 \qquad \Rightarrow 3 \times (k-3) \times \left(k - \frac{1}{3}\right) \le 0$$

$$\Rightarrow \frac{1}{3} \le k \le 3$$

Now since $k = \frac{x^{2} - x + 1}{x^{2} + x + 1}$, we have $\frac{1}{3} \le \frac{x^{2} - x + 1}{x^{2} + x + 1} \le 3$

Thus the minimum value of $\frac{x^2 - x + 1}{x^2 + x + 1}$ is $\frac{1}{3}$ and the maximum value of $\frac{x^2 - x + 1}{x^2 + x + 1}$ is 3.



Exercise

- 1. Would the expression, $5x 7 x^2$ have a maximum or a minimum value and what value would it be?
 - (1) Max, -7/2 (2) Max, 5/2 (3) Min, 5/2 (4) Max, -3/4
- 2. If the minimum value of $ax^2 bx + c$ is 0, find the necessary and sufficient condition between *a*, *b* and *c*.
 - (1) $a \times c < 0$ (2) $b^2 4ac < 0$ (3) $b^2 4ac = 0, a > 0$ (4) $b^2 4ac < 0, a < 0$
- 3. Find the maximum value of $\frac{1}{x^2 2x + 4}$
 - (1) 1 (2) 1/2 (3) 1/3 (4) ∞
- 4. Find the maximum value of $\frac{1}{x^2 4x + 1}$
 - (1) 2 (2) -1/3 (3) 1/3 (4) ∞
- 5. Find the minimum and maximum value of $\frac{x^2 3x + 4}{x^2 + 3x + 4}$
 - (1) 1/7, 7 (2) 1/3, 3 (3) 1/5, 5 (4) $\infty, -\infty$

6. How many integral values can the expression $\frac{15x^2 + 2x + 1}{x^2 - 2x - 1}$ NOT take?

(1) 0 (2) 1 (3) 5 (4) 7



Max Min

In questions of this type, usually the questions asks one to find the minimum among the maximum of two or more expressions or the maximum of the minimum of two or more expressions. Thus these questions tend to be confusing as the same question has maximum as well as minimum in the same sentence. To get a very effective grasp on these types of questions, it is advisable that you view the given expressions as graphs and then proceed. Let's take an example...

E.g. 58: If $y = \max(x-1, 5-2x)$, find the minimum value of y.

Consider $y = \max(x-1, 5-2x)$. This means that the value of y, for any value of x, will be the greater of the two: (x-1) or (5-2x).

Let's see what will be the value of the two expressions for various values of x and the graph thereof.

Considering the expression, (x - 1)...



In the graph above, we have plotted the *x* values and the corresponding (x - 1) values. We get a straight line.

Points to be observed in the above graph so that you do not spend any time in getting an idea of the graph, the next time you have to draw:



1. Since the expression is x - 1 i.e. a linear expression of the type ax + b, it has to be a straight line, as we have learnt in the topic on algebra basics.

2. Further, it has to cut the Y-axis at -1 since the *y*-intercept of the line ax + b is *b*. This also was learnt earlier.

3. Finally the line is an increasing line as its slope (slope of the line ax + b is *a*) is 1 i.e. positive. Also since the magnitude of the slope is 1, it has to be increase by a unit of 1 for every unit increase in *x*.

All of this is learnt earlier and hence one should have immediately plotted the line without the help of a table or of that of an axis.

Let's first make such deductions for the other expression (5 - 2x) and then actually plot the line and check if the conclusions were right.

1. Since the slope is -2, it will be a decreasing line. Further for every unit increase in *x*, the value of the expression will decrease by 2 units since the magnitude of slope is 2.

2. The *y*-intercept will be 5.

Plotting the values of (5 - 2x) for different values of $x \dots$

x	-2	-1	0	1	2	3	4	5
5 - 2r	9	7	5	3	1	_1	_3	_5



And the line is exactly as the observations we made.

So far we have seen how would (x - 1) and (5 - 2x) behave as x increases from $-\infty$ to ∞ . Now let us focus on the values of y for different values of x i.e. the graph of y.



For every value of x, we would get two values, each corresponding to (x - 1) and (5 - 2x). Of these two values, y would assume the higher one as the function is defined as y = maximum of (x - 1) or (5 - 2x). Thus, we can find the values of y for different values of x as done in the table below:

x	-2	-1	0	1	2	3	4	5
x - 1	-3	-2	-1	0	1	2	3	4
5 - 2x	9	7	5	3	1	-1	-3	-5
У	9	7	5	3	1	2	3	4

Let's see what this means on the graph...

Plotting the graph of both (x - 1) and (5 - 2x) on the same graph, for every *x*, we can read out two values on the *Y*-axis (vertical scale), one corresponding to x - 1 and other to 5 - 2x. These two values corresponds to the two points on the dotted line corresponding to each value of *x* e.g. for x = -1, the value read from the line corresponding to 5 - 2x is 7 and the value read from the line corresponding to x - 1 is -2. Similarly for any other value of *x*, one just needs to read the value on the vertical scale at the point where the dotted line intersects the two lines, x - 1 and 5 - 2x.

For each x, y would accept the higher value i.e. the point which is higher up because on the vertical scale point higher are greater than points lower down. Thus, in the figure below, the points which are circled are the values that y assumes. Check these values with the values of y as found in the above table and see for yourself that the two are same.



Now in the table and the graph we have just worked with integral values of x. But then x can assume any real value and thus the graph of y will be a continuous line and not just distinct point as shown. Joining all the distinct points representing the value of y for each corresponding value of x, the graph of y will look like the bold line in the graph below...





Thus, from the graph of y, we see that as x increases from $-\infty$ to ∞ , the value of y first decreases and then reaches a minimum point and starts increasing hereafter.

What will be the minimum value of *y*?

From the graph it can obviously be read to be 1 and it will occur when x = 2. What is so particular about this point? It is the intersection of the two lines, x - 1 and 5 - 2x. Thus the point can also be found by just equating x - 1 = 5 - 2x i.e. x = 2. Substituting x = 2 in either of x - 1 or 5 - 2x (they are the same for x = 2), we can get the *y* co-ordinate as 1.

While the above explanation was very very detailed and lengthy, the problem is one which can be solved in a jiffy once you are clear about the concept. Further you need not plot the lines perfectly on an *X*-*Y* plane. In an exam scenario, you should just think of the following:

E.g. 59: If $y = \max(x - 1, 5 - 2x)$, find the minimum value of y.

x - 1 is an increasing line as coefficient of x is positive

5-2x is a decreasing line as coefficient of x is negative

An increasing and decreasing line will intersect at some place. Imagining the frame where they intersect, the following figure can be imagined...



These lines have to be imagined without the aid of the axis or else you would be wasting a lot of time.



For any *x*, since *y* is the maximum of the two corresponding values, *y* will assume the higher up values and thus graph of *y* is the V formed above the intersection. The minimum value of *y* would occur at the intersection point i.e. for that *x* where x - 1 and 5 - 2x will be equal i.e. x - 1= 5 - 2x i.e. x = 2. For x = 2, the *y* co-ordinate will be 1. So the min value of *y* will be 1.

How would the graph of *y* change had *y* been defined as $y = \min(x - 1, 5 - 2x)$?

For all values of x, the values of x - 1 and 5 - 2x will remain the same. Its just that y will now take the minimum of the two values and thus the graph of y will be the bold line in the following figure...



Thus if the question would be the same i.e find the minimum value of y, the answer would be $-\infty$ because y is decreasing in both directions of x.

But in this scenario, a more logical question would be to find the maximum value of y. And the maximum value of y would again occur at the intersection i.e. at x = 2 and the maximum value would again be 1.

Do, not be in a hurry to deduce that the minimum or the maximum values of y would always occur at the intersection point. Look at the following example...

E.g. 60: If $y = \min(x - 2, 2x - 6)$, find the maximum value of y.

If, without giving the question its due thought, you just found the intersection point occurs when x - 2 = 2x - 6 i.e. at x = 4 and substituting x = 4, you found y = 2, you would arrive at a wrong answer.

In this case, both the line x - 2 and 2x - 6 are increasing line and thus the scenario need not be the same as discussed in the earlier example.

Though both the lines are increasing, the line representing 2x - 6 will increase at a faster rate that the line representing x - 2, because for every unit increase in x, 2x - 6 will increase by 2 whereas x - 2 will increase by 1. Thus even though both of them are increasing lines, they would intersect and the frame where they intersect will look like...





And since y assumes the minimum of the two values corresponding to each value of x, the graph of y would be the bold line in the figure below...



From the above it would be clear that y is an increasing function i.e. it will continuously increase as x increases. First it is increasing at a faster pace (slope = 2) and then (after $x > x_3$), it increases at a slower rate (slope = 1), but it increases nevertheless. Thus the maximum value of y would be ∞ . And the point of intersection does not have any major role to play in the answer. It is just the point where the slope of y changes.

If the function had been $y = \max(x - 2, 2x - 6)$, y would assume the higher points and its graph would be the dashed line in the above figure. It would still be an increasing line and thus its minimum and maximum value would be $-\infty$ and ∞ respectively.

When a range for x is given, we would get a specific answer to the question even when both lines being increasing or both a re decreasing....

If $y = \min(x - 2, 2x - 6)$, find the maximum and minimum value of y for $-10 \le x \le 10$.

In this case the graph would be exactly as discussed earlier but then since the range of x is limited between -10 and 10, the graph to be focused on exist only in this region and would just be the segment of the graph of y...





The graph of *y* is the bold line and thus it assumes a minimum value when x = -10 and a maximum value when x = 10. Finding these minimum and maximum value...

For
$$x = -10$$
, $y = \min(-10 - 2, 2 \times (-10) - 6) = \min(-12, -26) = -26$

For x = 10, $y = \min(10 - 2, 2 \times 10 - 6) = \min(8, 14) = 8$

Thus minimum value of y is -26 and maximum value of y is 8.

A common error beginners make is to mistake the maximum value of y as 14 because we are in search of maximum value and 14 is greater than 8. But this is wrong. 14 is not a value of the graph of y. Remember we are finding the maximum value of y, but y takes the minimum value of the two given expressions. Thus we are finding the maximum value after taking all the minimum values and the correct answer is 8.

Lastly, the expressions need not always be straight lines and could also be curves. Also there need not be just two expressions and there could be more also. For each of these case, its best first to get an idea of the graph of y and only on seeing the graph can you ascertain for which value of x would y be minimum or maximum.

E.g. 61: If $y = \min(x^2 - 4x, 6x - x^2)$, find the maximum value of y.

Let's take a look at the graphs of the expressions $x^2 - 4x$ and $6x - x^2$.

 $x^2 - 4x$ is a quadratic expression with a being positive. Thus the curve will be \bigcup shaped. The

minima of this curve will occur at $x = -\frac{b}{2a} = -\frac{-4}{2} = 2$

 $6x - x^2$ is also a quadratic expression with a being negative. Thus the curve will be \cap shaped. The maxima of this curve will occur at $x = -\frac{b}{2a} = -\frac{6}{2 \times (-1)} = 3$

Let's check if these curves would intersect each other or not. At the points of intersection, the values of the expressions would be equal to each other and hence equating the expressions..

 $x^2 - 4x = 6x - x^2$ i.e. $2x^2 - 10 x = 0$

i.e. $x^2 - 5x = 0$ i.e. x(x - 5) = 0 i.e. x = 0 or 5

Thus the two graphs would intersect at x = 0 and x = 5.

When x = 0, both the expressions would assume a value equal to 0 and when x = 5, both expressions would assume a value equal to $5^2 - 4 \times 5 = 5$.

Thus the graph would look like...





The graph of y is the bold line because for each value of x, it assumes the minimum values of the two expressions.

From the graph it should be obvious that the maximum value of y is 10 and it occurs when x = 5.

Exercise

If $y = \min(2x + 5, -x + 3)$, does y have a finite maximum or minimum value? And what is that finite 7. value? (4) Max, -2/3(1) Min, -2/3(2) Min, 11/3(3) Max, 11/3For what value of x does y assume the maximum value, y = min(4 - x, 0, 3x - 1)8. (4) $\frac{1}{3} \le x \le 4$ (2) 11/4(1) 5/4(3)0Find the maximum value of y, if $y = \max(-3x + 6, -5x + 6)$ and $-2 \le x \le 2$ 9. (1) - 4(2) 0(3) 12 (4) 1610. For what value of x would the expression $\min\left(\left|x-\frac{1}{2}\right|-\frac{1}{2},\frac{1}{2}-\left|x-\frac{1}{2}\right|\right)$ assume maximum value? (1) - 1/2(2) 0 or 1 (3) 1/2(4) 111. Find the least value of $\max(|x-3|, 5-|x|)$ (1)0(2) 1 (3)4(4) 5 12. For what value of x would y assume the maximum value, where $y = \min(x + 3, 2x - 1, 2 - x)$. (4) 2.5(1)7(2)1(3) - 2


Maxima / Minima based on $AM \ge GM$

Questions of this type essentially come in two variants:

- 1. Given a sum of positive variable, the maximum value of the product of the variables is asked
- 2. Given a product of positive variable, the minimum value of the sum of the variables is asked

Also all the following is limited to positive variables only.

If the question is as straight forward as: if a + b = 10, where a and b are positive real numbers, find the maximum value of $a \times b$, one can solve it just by using the following:

If the sum of positive variables is a constant, the product of the variables will be maximum when all of them are equal.

Thus the required maximum value of the product will occur when a = b = 5 and the maximum product will be 25. One can check this by just seeing the following pattern...

а	b	$a \times b$
0.1	9.9	0.99
1	9	9
2	8	16
3	7	21
4	6	24
4.5	5.5	24.75
5	5	25

and hereafter the product will start decreasing.

Similarly, if the product of positive variables is a constant, the sum of the variables will be minimum when all of them are equal.

But obviously problems cannot be just solved on this relation. It's better to understand the approach using $AM \ge GM$.

Part 1: Given a sum, maximum value of a product is asked.

E.g. 62: If *a*, *b* and *c* are positive variables and a + b + c = 12, find maximum value of $a \times b \times c$.

Consider the set *a*, *b*, *c*. Because AM of any set of numbers \geq GM of the set, we have, $\frac{a+b+c}{2} \geq \sqrt[3]{a \times b \times c}$

$$\Rightarrow 4 \ge \sqrt[3]{a \times b \times c} \Rightarrow a \times b \times c \le 64$$

The maximum value of $a \times b \times c$ will occur when the "equals to" sign comes into play in the above equality and this is when AM = GM, but this happens only when a = b = c.



Thus in short if a + b + c = 12, $a \times b \times c$ will be maximum when a = b = c = 4 each. Thus max value of $a \times b \times c$ is $4 \times 4 \times 4 = 64$.

E.g. 63: If *a*, *b* and *c* are positive variables and a + b + c = 12, find maximum value of $(a + 1) \times (b + 2) \times c$.

AM of
$$(a + 1)$$
, $(b + 2)$, $c \ge GM$ of $(a + 1)$, $(b + 2)$, c
$$\frac{(a+1)+(b+2)+c}{3} \ge \sqrt[3]{(a+1)\times(b+2)\times c}$$
$$\Rightarrow 5 \ge \sqrt[3]{(a+1)\times(b+2)\times c} \Rightarrow (a+1)\times(b+2)\times c \le 125$$

In all the problems that follow, the only challenge is to identify for what set of numbers should the AM and GM be compared. The set needs to be identified such that the AM should include the expression for which the sum is given and the GM should include the expression of which the product is asked.

E.g. 64: If a + 2b + 3c = 9 find maximum value of $a \times b \times c$. (Given a, b and c are positive)

AM of
$$a, 2b, 3c \ge GM$$
 of $a, 2b, 3c$

$$\frac{a+2b+3c}{3} \ge \sqrt[3]{a \times 2b \times 3c} \Rightarrow 3 \ge \sqrt[3]{a \times 2b \times 3c}$$
$$\Rightarrow a \times 2b \times 3c \le 27 \Rightarrow a \times b \times c \le \frac{27}{6}$$

E.g. 65: If a + b + c = 30, find maximum value of $a \times b^2 \times c^3$. (Given that *a*, *b* and *c* are positive).

If we consider the set *a*, *b*, *c*, the AM will include a + b + c but then the GM will have $a \times b \times c$ and not $a \times b^2 \times c^3$, the expression for which maximum value needs to be found.

For the GM to include $a \times b^2 \times c^3$, we need to consider the set *a*, *b*, *b*, *c*, *c*, *c*. But in this case the AM would include a + 2b + 3c, a sum which we do not know.

The way out is to consider the set: $a, \frac{b}{2}, \frac{b}{2}, \frac{c}{3}, \frac{c}{3}, \frac{c}{3}$. Now the AM will include

a + b + c, a sum we know and the GM will have $a \times b^2 \times c^3$, the product we need to find maximum value of.



$$AM \text{ of } a, \frac{b}{2}, \frac{b}{2}, \frac{c}{3}, \frac{c}{3}, \frac{c}{3} \ge GM \text{ of } a, \frac{b}{2}, \frac{b}{2}, \frac{c}{3}, \frac{c}{3}, \frac{c}{3}$$
$$\Rightarrow \frac{a + \frac{b}{2} + \frac{b}{2} + \frac{c}{3} + \frac{c}{3} + \frac{c}{3}}{6} \ge \sqrt{a \times \frac{b}{2} \times \frac{b}{2} \times \frac{c}{3} \times \frac{c}{3} \times \frac{c}{3}}$$
$$\Rightarrow \frac{a + b + c}{6} \ge \sqrt{\frac{a \times b^2 \times c^3}{2^2 \times 3^3}} \Rightarrow 5 \ge \sqrt{\frac{a \times b^2 \times c^3}{2^2 \times 3^3}}$$
$$\Rightarrow \frac{a \times b^2 \times c^3}{2^2 \times 3^3} \le 5^6 \Rightarrow a \times b^2 \times c^3 \le 2^2 \times 3^3 \times 5^6$$

Thus maximum value of $a \times b^2 \times c^3$ is $2^2 \times 3^3 \times 5^6$.

E.g. 66: If a + 2b + 3c = 63, find maximum value of $a^3 \times b^5 \times c$. (Given *a*, *b* and *c* are positive)

Considering the set $\frac{a}{3}$, $\frac{a}{3}$, $\frac{a}{3}$, $\frac{2b}{5}$, $\frac{2b}{5}$, $\frac{2b}{5}$, $\frac{2b}{5}$, $\frac{2b}{5}$, 3c and because for this set

$$AM \ge GM,$$

$$\Rightarrow \frac{a+2b+3c}{9} \ge \sqrt[9]{a^2 \times b^2 \times c \times \frac{2^5 \times 3}{3^3 \times 5^5}} \Rightarrow \frac{7^9 \times 3^2 \times 5^5}{2^5} \ge a^3 \times b^5 \times c$$

Part 2: Given product and minimum sum asked

The above approach of using AM and GM is advisable because then using the same funda, one can even solve all the problems asked in reverse way i.e. product is given and minimum sum is asked.

E.g. 67: If a, b, c are positive and $a \times b \times c = 216$, find minimum value of a + b + c.

8

$$\frac{a+b+c}{3} \ge \sqrt[3]{a \times b \times c}$$
$$\Rightarrow \frac{a+b+c}{3} \ge 6 \quad \Rightarrow a+b+c \ge 1$$

E.g. 68: If *a* and *b* are positive and $a \times b = 6$, find the minimum value of 2a + 3b.

AM of 2a and $3b \ge$ GM of 2a and 3b

$$\frac{2a+3b}{2} \ge \sqrt{2a \times 3b} \quad \Rightarrow 2a+3b \ge 12$$

E.g. 69: If a and b are positive and $a^3 \times b^2 = 108$, find the minimum value of a + b.

If we consider AM and GM of *a* and *b*, we will have the expression a + b (in AM), but we would also need $a \times b$, which we do not know.

If we consider AM and GM of a^3 and b^2 , we will have $a^3 \times b^2$ (in GM), but in finding AM we will get $a^3 + b^2$.



The solution is to consider the AM and GM of $\frac{a}{3}$, $\frac{a}{3}$, $\frac{a}{3}$, $\frac{b}{2}$, $\frac{b}{2}$. Now the GM will boil

down to $\sqrt[5]{\frac{a^3 \times b^2}{3^3 \times 2^2}}$ and we know the value of this. Also the AM will boil down to $\frac{a+b}{5}$ and we can find the minimum value of a + b. AM of $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}, \frac{b}{2}, \frac{b}{2} \ge$ GM of $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}, \frac{b}{2}, \frac{b}{2}$ $\Rightarrow \frac{a+b}{5} \ge \sqrt[5]{\frac{a^3 \times b^2}{3^2 \times 2^2}} \Rightarrow \frac{a+b}{5} \ge 1 \Rightarrow a+b \ge 5$. Thus minimum value is 5.

E.g. 70: If $a^4 \times b^2 = 2^4 \times 3^2 \times 5^4$, find minimum value of 3a + 5b, given *a* and *b* are positive.

$$AM \text{ of } \frac{3a}{4}, \frac{3a}{4}, \frac{3a}{4}, \frac{3a}{4}, \frac{5b}{2}, \frac{5b}{2} \ge GM \text{ of } \frac{3a}{4}, \frac{3a}{4}, \frac{3a}{4}, \frac{3a}{4}, \frac{5b}{2}, \frac{5b}{2}$$
$$\Rightarrow \frac{3a+5b}{6} \ge \sqrt[6]{a^4 \times b^2 \times \frac{3^4 \times 5^2}{2^{10}}}$$
$$\Rightarrow \frac{3a+5b}{6} \ge \frac{3\times5}{2} \quad \Rightarrow 3a+5b \ge 45$$

Exercise

Note for questions 13 to 24: *a*, *b* and *c* are positive real numbers only.

13. If a + b + c = 12, what is the maximum value of $a^2 \times b \times c$?

(1) 3 (2) 64 (3) 81 (4) 324

14. If a + 2b + 3c = 12, what is the maximum value of $a \times b^2 \times c^3$?

(1) 64 (2) 81 (3) 1296 (4) 6912

15. If 2a + 3b + 5c = 12, what is the maximum value of $a \times b \times c$?

(1) 64 (2) 32 (3) 32/15 (4) 64/15

- 16. If a + 2b + 3c = 12, what is the maximum value of $a^2 \times b \times c$?
 - (1) 162 (2) 81 (3) 54 (4) 27
- 17. If $a \times b \times c = 4.5$, what is the minimum value of 2a + b + 3c?
 - (1) 3 (2) 9 (3) 27 (4) 81



18. If $a^2 \times b \times c = 16$, what is the minimum value of 2a + b + c?

19. If $a^2 \times b^3 \times c = 108$, what is the minimum value of a + b + c?

20. If $a^2 \times b^3 \times c = \frac{2^8 \times 3^7}{5}$, what is the minimum value of 3a + b + 5c?

(1) 36 (2) 30 (3) 24 (4) 6

21. If *a* and *b* are positive variables, what is the least value of $\frac{a}{b} + \frac{b}{a}$?

$$(1) -2 (2) 0 (3) 2 (4) 1$$

22. If *p*, *q*, *r* are positive variables what is the least value of $\frac{p}{q} + \frac{q}{r} + \frac{r}{p}$?

(1) 1/3 (2) 1 (3) 3 (4) 9

23. If *p*, *q*, *r* are positive variables, what is the least value of $(p+q+r) \times \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)$?

(1) 1/3 (2) 1 (3) 3 (4) 9

24. If $a \times b \times c \times d = 1$, where *a*, *b*, *c* and *d* are positive variables, find the least value of (1 + a)(1 + b)(1 + c)(1 + d)?

- (1) 1 (2) 2 (3) 4 (4) 16
- 25. Which of the following is smaller: 51^{101} or 101!?
 - (1) 51^{101} (2) 101! (3) Both are equal



Function is a term used to denote dependency. Thus when we say y = f(x), we just mean to say that the value of y will depend on the value of x. To specify exactly how is the value of y dependent on x, the user needs to define the function further. Thus, only when the user defines y

as
$$y = f(x) = 3x - 4$$
 or $y = f(x) = x^2 - 5x + 6$ or $y = f(x) = x + \frac{1}{x}$ or $f(\theta) = \sin \theta$ any other relation,

is the dependency more clear. Please be clear that the relation could be anything that the user desires and thus the function will be clearly specified in the question itself.

Usually the 'y' is dropped from the above relations and it is just expressed as f(x) = 3x - 4 and similarly for other relations. Also there is nothing specific that the letter 'f' be used to denote functions. Thus we could also write the above dependency as g(x) = 3x - 4 or $h(x) = x + \frac{1}{x}$

Consider a relation to be defined as $f(x) = \frac{x+1}{x}$ for all x not equal to 0.

The value of f(x) is dependent on the value that x assumes. x here is a variable and could assume any value. Thus as x changes, the value of f(x) would also change.

f(-2) implies the value of the function f(x) when x = -2 and is obtained by substituting x by -2, wherever it appears in the relation $f(x) = \frac{x+1}{x}$. Substituting x as -2 in (i), we get f(-2) =

$$\frac{-2+1}{-2} = \frac{-1}{-2} = \frac{1}{2} \text{ or } 0.5$$

f(3) implies the value of the function f(x) when x = 3 and is obtained by substituting x by 3, wherever it appears in the relation $f(x) = \frac{x+1}{x}$

Substituting *x* as 3 in (i), we get $f(3) = \frac{3+1}{3} = \frac{4}{3} = 1.33...$

By the same logic as above, $f\left(\frac{1}{2}\right)$ implies the value of the function f(x) when $x = \frac{1}{2}$ and is obtained by substituting x by $\frac{1}{2}$, wherever it appears in the relation $f(x) = \frac{x+1}{x}$.

Substituting x as $\frac{1}{2}$ in (i), we get $f\left(\frac{1}{2}\right) = \frac{\frac{1}{2} + 1}{\frac{1}{2}} = \frac{\frac{3}{2}}{\frac{1}{2}} = 3$



Going by the above logic, if we write $\frac{1}{x^2}$ in place of x in the relation $f(x) = \frac{x+1}{x}$, we would

get
$$f\left(\frac{1}{x^2}\right) = \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2}} = \frac{\frac{1 + x^2}{x^2}}{\frac{1}{x^2}} = 1 + x^2$$
. This is entirely valid operation and to understand this, the

concept of function as an operator, as explained below, is a very handy.

Function as an Operator

Any defined function can also be looked as an operator, anything that performs a given operation, say a black box which is programmed to do the calculation as defined by the user.

Thus if the function is defined as done in the above examples i.e. $f(x) = \frac{x+1}{x}$, the box will be

programmed to calculate $\frac{x+1}{x}$ for any input x. The utility of this concept is that it can facilitate in thinking x as to just represent any input

in thinking *x* as to just represent any input....



$$Input = x \longrightarrow \frac{input + 1}{input} = \frac{x + 1}{x} \longrightarrow Output = \frac{x + 1}{x}$$

If Input = 2

$$Input = 2 \longrightarrow \frac{Input + 1}{Input} = \frac{2 + 1}{2} \longrightarrow Output = \frac{2 + 1}{2} = \frac{3}{2}$$

What this approach helps in, is when the input is not a numerical value per se, but is itself an expression. E.g. continuing with the same operator *f* which calculates $\frac{input+1}{input}$ for any input, the following will be the situation when the input itself is some expression in *x*.



If
$$Input = \frac{1}{x^2}$$

 $Input = \frac{1}{x^2}$
 $input + 1 = \frac{\frac{1}{x^2} + 1}{\frac{1}{x^2}}$
 $Output = 1 + x^2$

If Input = 3x - 1

$$Input = 3x - 1 \longrightarrow \boxed{\frac{input + 1}{input} = \frac{(3x - 1) + 1}{(3x - 1)}} \longrightarrow Output = \frac{3x}{3x - 1}$$

E.g. 71: If
$$f(x) = \frac{2x+1}{2-3x}$$
, find the value of $f\left(-\frac{1}{6}\right)$.

$$f\left(-\frac{1}{6}\right) = \frac{2 \times \left(-\frac{1}{6}\right) + 1}{2-3 \times \left(-\frac{1}{6}\right)} = \frac{-\frac{1}{3} + 1}{3+\frac{1}{2}} = \frac{\frac{2}{3}}{\frac{7}{2}} = \frac{4}{21}$$

E.g. 72: If $f(x) = x^2 + 3x + 4$, find the expression for f(x - 3).

Replacing x by (x - 3) in the expression $f(x) = x^2 + 3x + 4$, we get

$$f(x-3) = (x-3)^2 + 3(x-3) + 4 = x^2 - 6x + 9 + 3x - 9 + 4 = x^2 - 3x + 4.$$

E.g. 73: If f(x - 3) = x + 3, find the value of f(7)

Please note that in this case, the function is defined for (x - 3). Thus if we need to find the value of f(7), we need to replace x by 10 and not by 7. Replacing x by 10, we get f(10 - 3) = 10 + 3 i.e. f(7) = 13.

Functions need not be dependent on just one variable. A function could be dependent on two or more than two variables as well...

E.g. 74: If $f(x, y) = x^2 - xy + y^2$, find the value of f(2, -3).

Replacing x by 2 and y by -3, we get, $f(2, -3) = 2^2 - 2 \times (-3) + (-3)^2 = 4 + 6 - 9 = 1$

E.g. 75: If $@(x, y) = \frac{x}{y} + \frac{y}{x}$, for what value of y would f(2, y) be equal to $\frac{13}{6}$?

From the data given, $\frac{2}{y} + \frac{y}{2} = \frac{13}{6}$



Multiplying throughout by 6y, we get $12 + 3y^2 = 13y$

$$\Rightarrow 3y^2 - 13y + 12 = 0$$
$$\Rightarrow 3 \times (y - 3) \times \left(y - \frac{4}{3}\right) = 0 \Rightarrow y = 3 \text{ or } \frac{4}{3}$$

Composite Functions

Consider two functions defined as $f(x) = x^2 - 1$ and g(x) = 2x + 1

We can define many more operations by clubbing the two functions together e.g. f(g(x)), g(f(x)), f(f(g(x))), etc. Such functions which are two more nested functions are called as composite functions. Let's take f(g(x)) and understand what they mean...

In the operation f(g((x)), x is the independent variable, or x is the input. The operator g is first performed on x, the input and the result is denoted as g(x). Now the operator f is performed on this g(x) i.e. the output of the operator g is acting as the input to the operator f(x) and the end result is denoted as f(g(x)). Pictorially this means...



Thus as seen in the visual above...

 $f(g(x)) = f(2x - 1) = (2x - 1)^2 + 1 = 4x^2 - 4x$

If we had to compute g(f(x)), first the operator, f, would have been performed on x and then the operator, g, would have been performed on f(x). Thus, we would have, $g(f(x)) = g(x^2 - 1) = 2(x^2 - 1) + 1$

Please note that $f(g(x)) \neq g(f(x))$.

E.g. 76: If $f(x) = \frac{1}{x-1}$ and g(x) = x + 1, find the value of f(g(x)) and also the value of g(f(x)).

$$f(g(x)) = f(x+1)$$

Replacing x by
$$(x + 1)$$
 in $f(x) = \frac{1}{x-1}$, we get $f(x + 1) = \frac{1}{(x+1)-1} = \frac{1}{x}$



Thus
$$f(g(x)) = \frac{1}{x}$$

 $g(f(x)) = g\left(\frac{1}{x-1}\right)$. Replacing x by $\frac{1}{x-1}$ in $g(x) = x+1$, we get
 $g\left(\frac{1}{x-1}\right) = \frac{1}{x-1} + 1 = \frac{x}{x-1}$
Thus, $g(f(x)) = \frac{x}{x-1}$

E.g. 77: If f(x) = 2x + 3 and g(x) = 3x + 2, for what value of x would f(g(f(x))) = g(f(g(x)))

Finding f(g(f(x)))... g(f(x)) = g(2x + 3). Since the operator g computes $3 \times (input) + 2$, we have $g(2x + 3) = 3 \times (2x + 3) + 2 = 6x + 11$ $f(g(f(x))) = f(6x + 11) = 2 \times (6x + 11) + 3 = 12x + 25$ Similarly we can find the value of g(f(g(x)))... f(g(x)) = f(3x + 2). Since the operator f computes $2 \times (input) + 3$, we have $f(3x + 2) = 2 \times (3x + 2) + 3 = 6x + 7$ $g(f(g(x))) = g(6x + 7) = 3 \times (6x + 7) + 2 = 18x + 23$ Equating f(g(f(x))) = g(f(g(x))), we have 12x + 25 = 18x + 23 i.e. 6x = 2 i.e. $x = \frac{1}{3}$

Iterative function:

Iterative function is a specific type of composite function where the function is performed on the output of the same function itself i.e. we compute f(f(x)) or f(f(f(x))) or any such series

of fs

E.g. 78: If f(x) = 3x - 2, find the value of f(f(f(2)))





E.g. 79: If
$$f(x) = \frac{x-1}{x+1}$$
, find the expression for $f(f(x))$ and for $f(f(x))$).
$$f(x) = \frac{x-1}{x+1} \quad \dots \dots (i)$$

Replacing x by f(x) in (i), we get $f(f(x)) = \frac{f(x)-1}{f(x)+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{\frac{-2}{x+1}}{\frac{2x}{x+1}} = -\frac{1}{x}$

Thus, $f(f(x)) = -\frac{1}{x}$

Replacing x by f(f(x)), in (i), we get

$$f(f(f(x))) = \frac{f(f(x)) - 1}{f(f(x)) + 1} = \frac{-\frac{1}{x} - 1}{-\frac{1}{x} + 1} = \frac{-\frac{1 - x}{x}}{-\frac{1 - x}{x}} = \frac{x + 1}{x - 1}$$

Thus, $f(f(f(x))) = \frac{x+1}{x-1}$

It's difficult to pronounce and to write expressions like f(f(x)) and thus we use a short-

hand notation for such longish expressions. The short-hand notation for f(f(f(x))) is $f^3(x)$. Please note that the 3 is not the index or power of f(x). It is a superscript and is read as 'f three of x'. To denote the third power of f(x), we use $(f(x))^3$

When you use a notation like $f^3(x)$, first it has to be defined so that the user also understands it as f(f(f(x))). For this the function f(x) is denoted as $f^1(x)$ and then the iterative nature is expressed as: $f^n(x) = f^1(f^{n-1}(x))$

Putting n = 2, we get $f^2(x) = f^1(f^1(x))$

Putting n = 3, we get $f^3(x) = f^1(f^2(x))$, and putting the value of $f^2(x)$ just found, $f^3(x) = f^1(f^1(f^1(x)))$

Continuing in the same manner, we can find the iterative function to any degree of nested functions and simply put the $f^{n}(x)$ refers to the function being performed *n* times, iteratively.



E.g. 80: If $f^{1}(x) = x^{2}$, find the value of $f^{10}(x)$, where $f^{n}(x) = f^{1}(f^{n-1}(x))$.

$$f^{2}(x) = f^{1}(f^{1}(x)) = f^{1}(x^{2}) = (x^{2})^{2} = x^{4}$$
$$f^{3}(x) = f^{1}(f^{2}(x)) = f^{1}(x^{4}) = (x^{4})^{2} = x^{8}$$

Similarly we can check that we will get $f^4(x) = x^{16}$ and thus we can generalize and conclude that $f^{10}(x) = x^{1024}$

Exercise

1. If $f(x, y, z) = \max(\min(x, y), \min(y, z), \min(x, z))$, find f(2, 3, 4). (1)1(2) 2(3)3(4) 42. If $f(x + 2, y - 3) = (x - 3) \times (y + 2)$, find the value of f(-2, 3). (2) - 25(1) - 56(3) - 6(4) - 243. f(x) = x - 1 and $g(x) = \frac{1}{x+1}$. Find f(g(x)) and g(f(x)). (1) $\frac{x}{x-1}, \frac{1}{x}$ (2) $\frac{-x}{x+1}, \frac{1}{x}$ (3) $\frac{x}{1-x}, \frac{1}{x}$ (4) $\frac{x}{1-x}, x$ 4. $f(x) = \frac{x+1}{x-1}$ and $g(x) = \frac{1-x}{1+x}$. For what value of x is f(g(x)) = g(f(x))? (1)0(2)1(3) 2(4) All except 0 **Directions for Questions 10 to 14:** $f^{n}(x) = f^{1}(f^{n-1}(x))$ 5. If $f^{1}(x) = x - 1$, find $f^{10}(x)$ (1) x - 1 (2) x - 9(3) x - 10(4) x - 116. If $f^{1}(x) = 1 - \frac{1}{r}, f^{75}(x)$ (2) $\frac{1}{1-r}$ (3) $\frac{1}{r-1}$ (4) $1-\frac{1}{r}$ (1) x7. If $f^{1}(x) = \frac{x}{x-1}$, find $f^{100}(x)$ (2) $\frac{x}{x-1}$ (3) $\frac{1}{x-1}$ (4) $\frac{x}{r+1}$ (1) x



- 8. If $f^{1}(t) = (1-t^{2})^{\frac{1}{2}}$, find $f^{100}(t)$ (1) $\sqrt{1-t^{2}}$ (2) $\sqrt{t^{2}-1}$ (3) t (4) $\sqrt[4]{1-t^{4}}$ 9. If $f^{1}(a) = 2a + 1$, find $f^{n}(a)$. (1) $2^{n-1} \times a + (2^{n-1}-1)$ (2) $2^{n} \times a + (2^{n}-1)$ (3) $2^{n+1} \times a + (2^{n+1}-1)$ (4) $2^{n} \times a + 2^{n} + 1$
- 10. If $@(x, y) = average of x and y, #(x, y) = x \times y, $(x, y) = x/y, which of the following expression represents the average of x, y and z$
 - (1) @(@(x, y), z)(2) #(@(#(@(x, y), 2), z), 2)(3) #(\$(@(#(@(x, y), 2), z), 3), 2)(4) \$(@(#(@(x, y), 2), z), 3)

Even and Odd Functions

Even Functions are those functions for which the value of f(x) and f(-x) are same for all real values of x i.e. f(x) = f(-x) for all x.

E.g. $f(x) = x^2$ or f(x) = |x| or $f(x) = 3x^2 + 1$

Consider $f(x) = x^2$. Replacing x by -x, we get $f(-x) = (-x)^2 = x^2 = f(x)$, for all x. And the same is true for any expression which has only even powers of x.

Odd Functions are those functions for which the value of f(-x) is the negative of the value of f(x) for all real values of x i.e. f(-x) = -f(x) for x.

E.g.
$$f(x) = x^3$$
 of $f(x) = \frac{1}{x}$ or $f(x) = x^3 - x$

Consider $f(x) = x^3 - x$.

$$f(2) = 2^3 - 2 = 6$$
 and $f(-2) = (-2)^3 - (-2) = -8 + 2 = -6$. Thus $f(-2) = -f(2)$

And this will be true for any value of x that you choose. This can also be verified by the general case as follows: $f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x)$

Please note that most of the functions are neither even nor odd. Only a few of the functions can be classified as even or odd. E.g. for the function $f(x) = x^2 + x$, f(2) would be $2^2 + 2 = 6$ and

 $f(-2) = (-2)^2 + (-2) = 4 - 2 = 2$, which is neither equal to f(2) nor equal to -f(2).

In entrance exams, particularly, CAT, questions based on concept of Even and Odd function appear in the form of graphs being given and the question is to identify if it is an even or an odd function. So let's have a look at how would graphs of Even and Odd function look like...



Graph of Even Functions:

Consider any even function, f(x). Thus we will have f(x) = f(-x) for all x. In the figure below, say A is a point that lies on the graph of f(x) corresponding to x = 2 and B is also a point on the graph of f(x) that corresponds to x = -5.

Since f(-2) = f(2), the *y* co-ordinate of the point on the graph corresponding to x = -2 should be the same as the *y* co-ordinate of *A*. Thus, there would be a point *A*' on the graph which would be the reflection of *A* across the *Y*-axis, as shown in figure (ii). The reflection is across the *Y*-axis, because the value of the *y* co-ordinate, viz f(2) and f(-2) is the same whereas the *x* co-ordinate changes from 2 to -2.

If we start with x = -5, i.e. with the point *B* assumed to lie on the graph, then -x would be 5 and since f(-5) = f(5), the point *B* must also be reflected across the *Y*-axis and the point *B*', shown in figure (ii), should also lie on the graph of f(x).



And the above reasoning will be valid for all points on the graph since the relation f(-x) = f(x) is valid for all values of x.

Thus, essentially, any point on the graph will also have its reflection across the *Y*-axis lying on the graph and thus, the graph will be symmetric across the *Y*-axis.

Let's take a couple of examples and check if the graph is indeed a symmetric about the Y-axis...



So, we see that graphs of Even Functions are symmetric about the Y-axis.



Graphs of Odd Functions

Consider any odd function, f(x). Thus we will have f(-x) = -f(x) for all x. In the figure below, say A is a point that lies on the graph of f(x) corresponding to x = 2 and B is also a point on the graph of f(x) that corresponds to x = -5.

Since f(-2) = -f(2), the *y* co-ordinate of the point on the graph corresponding to x = -2 should be the negative of the *y* co-ordinate of *A*. Thus, there would be a point *A*' on the graph which would be the reflection of *A*, across the *Y*-axis and the *X*-axis, as shown in figure (ii). Since *x* changes from 2 to -2, a reflection across *Y*-axis is necessitated and since the *y*-coordinates i.e. f(2) and f(-2), are negatives of each other, a reflection across *X*-axis is also necessitated.

If we start with x = -5, i.e. with the point *B* assumed to lie on the graph, then -x would be 5 and since f(-5) = -f(5), the point *B* must also be reflected across the *Y*-axis and the *X*-axis and the point *B*', shown in figure (ii), should also lie on the graph of f(x).



The above reasoning will be valid for all points on the graph since the relation f(-x) = -f(x) is valid for all values of *x*.

Thus, essentially, any point on the graph will also have its reflection across the *X*-axis and the *Y*-axis lying on the graph and thus, the graph will be symmetric across the origin. Or more specifically the 1^{st} and the third quadrant will be reflections of each other and so would the 2^{nd} and 4^{th} quadrant.

Let's take a couple of examples and check if the graph has the 1^{st} and 3^{rd} quadrants as reflections of each other and the 2^{nd} and 4^{th} are reflections of each other...



In the above, we saw how the concept of even and odd functions dictates the graph of the function. A similar concept as the above can also be used to relate two different graphs, say f(x) and g(x).

Case I: f(x) = g(-x)

Say if f(x) = g(-x), then every point on the graph of f(x) will have its reflection across the *Y*-axis as points on the graph of g(x), and obviously vice-versa would also be true i.e. every point on the graph of g(x) will have its reflection across the *Y*-axis as points on the graph of f(x).

If the above statement is not clear, have a look at the following figure. Consider point *A* to be a point on the graph of f(x) corresponding to x = 2. Since g(-2) = f(2), we should also have a point *A*' with x = -2 (i.e. a reflection across *Y*-axis) and the *y*-coordinate of *A*' should be same as that of *A* since g(-2) is same as f(2).

Alternately, if we start with a point *B* lying on the graph of g(x) for x = -5, then since f(5) = g(-5), the reflection of point *B* across the *Y*-axis i.e. with x = 5 and *y* coordinate same as that of *B*, shown as *B*' in the figure, has to lies on the graph of f(x).



Thus for the relation f(x) = g(-x), the two graph are going to be the 'mirror' images of each other. Its called mirror image since either graph is the reflection of the other graph, as if a mirror is placed vertically between the two graphs...





Case II: f(x) = -g(x)

Say if f(x) = -g(x), then every point on the graph of f(x) will have its reflection across the *X*-axis as points on the graph of g(x), and obviously vice-versa would also be true i.e. every point on the graph of g(x) will have its reflection across the *X*-axis as points on the graph of f(x).

If the above statement is not clear, have a look at the following figure. Consider point *A* to be a point on the graph of f(x) corresponding to x = 2. Since f(2) = -g(2) or g(2) = -f(2), we should have a point *A*' with x = 2 and the *y*-coordinate of *A*' should be negative of the *y* coordinate of *A* (i.e. reflection across *X*-axis) because g(2) = -f(2).

Alternately, if we start with a point *B* lying on the graph of g(x) for x = -5, then since f(5) = -g(5), the reflection of point *B* across the *X*-axis i.e. with *x* being same as 5 but *y* coordinate being negative of the *y* coordinate of *B*, shown as *B*' in the figure, has to lies on the graph of f(x).



Thus for the relation f(x) = -g(x), the two graph are going to be the 'water' images of each other. Its called water image since either graph is the reflection of the other graph across the *X*-axis as if it is an image viewed downwards on a water surface...



Please note that two functions, f(x) and g(x) could satisfy more than one relation simultaneously, e.g. consider the following figure, in which f(x) and g(x) are the mirror images of each other and at the same time are the water images as well i.e. if the graphs of f(x) and g(x) are as given below then both the relations f(x) = g(-x) and f(x) = -g(x) are valid.



Case III: f(x) = -g(-x)

Say if f(x) = -g(-x), then every point on the graph of f(x) will have its reflection across the *X*-axis and the *Y*-axis as points on the graph of g(x), and obviously vice-versa would also be true i.e. every point on the graph of g(x) will have its reflection across the *X*-axis and the *Y*-axis as points on the graph of f(x).

If the above statement is not clear, have a look at the following figure. Consider point *A* to be a point on the graph of f(x) corresponding to x = 2. Since f(2) = -g(-2), we should have a point *A*' with x = -2 (reflection across the *Y*-axis) and the *y*-coordinate of *A*' should also be the negative of the *y* coordinate of *A* (i.e. reflection across *X*-axis) because g(2) = -f(2).

Alternately, if we start with a point *B* lying on the graph of g(x) for x = -5, then since f(-5) = -g(5), the reflection of point *B* across the *Y*-axis i.e. with *x* changing from 5 to -5 and *y* coordinate being negative of the *y* coordinate of *B*, shown as *B*' in the figure, has to lies on the graph of f(x).



This is a case of mirror and water image i.e the 1st quadrant of f(x) and 3rd quadrant of g(x) will be reflections of each other; 2nd quadrant of f(x) and 4th quadrant of g(x) will be reflection of each other; 3rd quadrant of f(x) and 1st quadrant of g(x) will be reflections of each other; and 4th quadrant of f(x) and 2nd of g(x) will also be reflections of each other...



Greatest Integer Function

A specific function used often is the Greatest Integer Function and is denoted by [x]

Thus, f(x) = [x] refers to the greatest integer function and is defined as the greatest integer less than or equal to *x*.

Thus, we are looking for integers less than or equal to x and from among these integers, the greatest of them is the value that the function returns.

When we are finding integers less than or equal to x, we are looking for integers on the left of x and the greatest of them would be the first integer that is encountered while moving leftwards from x...



Thus [4.5] = 4, [7.002] = 7, [15.997] = 15, [12] = 12

Be very careful when finding the Greatest Integer Function of negative numbers....

[-3.5] = -4, [-10.01] = -11, [-3.99] = -4, [-5] = -5.

The greatest integer function finds its use in situations where we are interested in just the quotient of a division. Thus, if we want to denote just the integer part when 12 is divided by 5,

we can denote it as
$$\left[\frac{12}{5}\right]$$
 because $\left[\frac{12}{5}\right] = \left[2.4\right] = 2$



Exercise

(1) f(x) = g(-x);

11. Select the correct relation between f(x) & g(x)



12. Select the correct relation between f(x) & g(x)



I. f(x) = g(-x)	II. $f(x) = -g(x)$	III. $f(x) = -g(-x)$	
(1) I	(2) II	(3) I and II	(4) All three

Directions for questions 13 to 15: [*x*] represents the greatest integer less than or equal to *x*. And $\{x\} = x - [x]$

- 13. Which of the following statement is true:
 - (1) $[2x] = 2 \times [x]$ is true for all values of *x*.

(2) $[2x] = 2 \times [x]$ is true ONLY for integral values of *x*.

(3) $[2x] = 2 \times [x]$ is false for ALL negative integral values of x.

(2) f(x) = -g(x);

- (4) $[2x] = 2 \times [x]$ is false for positive values of x having the decimal more than or equal to 0.5
- 14. Which of the following statement is true:
 - (1) [2x + 2y] = [x] + [y] + [x + y] is true for all values of x and y
 - (2) [2x + 2y] = [x] + [y] + [x + y] is true ONLY for integral values of x and y
 - (3) [2x+2y] = [x] + [y] + [x+y] is true for all positive values of x and y such that x + y = [x+y]
 - (4) [2x + 2y] = [x] + [y] + [x + y] is NEVER true for negative values of x and y
- 15. Find the sum of first 50 terms of $\left[\frac{1}{5}+1\right]+\left[\frac{2}{5}+2\right]+\left[\frac{3}{5}+3\right]+\left[\frac{4}{5}+4\right]+\dots$
 - (1) 1510 (2) 1500 (3) 1275 (4) 1300





Progressions are series of numbers, where each successive number in the series is derived from the previous number according to a rule. Two most popular progressions are Arithmetic Progression (A.P.) and Geometric Progression (G.P.) Lately the entrance exams have also had questions on Arithmetico-Geometric Progressions (AGP).

Arithmetic Progression

In an *A.P.*, the successive term is got by adding a constant value to the previous term. Each *A.P.* is characterized by its first term and by the constant that is added e.g. if the first term is 15 and the constant to be added is 4, we get the following series: 15, 19, 23, 27, 31, 35, 39,

Thus we see that the difference between the successive terms of an A.P. is a constant (the constant that is added)

The first term of an A.P. is denoted by a and the constant difference between the successive terms is called 'common difference' and is denoted by d.

Finding the value of the n^{th} term:

The value of any n^{th} term is denoted by T_n and one can find the value of any n^{th} term using the formula:

$$T_n = a + (n-1)d$$

E.g. 81: What is the 25th term of the series -23, -16, -9, -2, 5, 12...

Here a = -23 and d = 7 and we need to find the 25th term i.e. T_{25} . Thus substituting n = 25 in the formula for T_{n} , we get $T_{25} = -23 + 24 \times 7 = -23 + 168 = 145$.

Finding the number of terms in an given A.P.:

Given any series, the above formula can be re-arranged to find the number of terms as follows:

$$n = \frac{\text{Last Term} - \text{First Term}}{d} + 1$$

E.g. 82: Find the number of terms in the series: 32, 29, 26, 23,, -19.

This is an *A*.*P*. with the first term being 32, last term being -19 and d = -3.

Thus the number of terms =
$$\frac{-19 - 32}{-3} + 1 = \frac{-51}{-3} + 1 = 18$$



Finding the sum of *n* terms of an A.P.:

The only other formula used in A.P. is the formula to find the sum of first n terms of an A.P.

$$S_n = \frac{n}{2} \{ 2a + (n-1)d \}$$
 or $S_n = \frac{n}{2} ($ First Term + Last Term $)$

If the first and the last term is known you can use the second formula, but you will yet need to know the number of terms.

E.g. 83: Find the sum of first 10 terms of the series 11, 21, 31,

This is an *A*.*P*. with a = 11 and d = 10 and we need to find S_{10} . Thus putting the value of n = 10 in the formula for the sum of first *n* terms of *A*.*P*, we get,

$$S_{10} = \frac{10}{2} (2 \times 11 + 9 \times 10)$$
 $S_{10} = 5 (22 + 90) = 5 \times 112 = 560$

E.g. 84: Find the sum of all terms of the series $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$,, $\frac{31}{2}$

If we find the difference between any two terms we see that it is always $\frac{1}{2}$. Thus the series given is an *A.P.* Since we know the first term and the last term, let's find the number of terms using the formula $S_n = \frac{n}{2}$ (First Term + Last Term). To use this we would need to know the number of terms i.e. *n*.

The number of terms =
$$\frac{\text{Last Term} - \text{First Term}}{d} + 1 = \frac{\frac{31}{2} - \frac{1}{2}}{\frac{1}{2}} + 1 = \frac{\frac{30}{2}}{\frac{1}{2}} + 1 = 31$$

The sum of the terms $=\frac{31}{2}\left(\frac{1}{2}+\frac{31}{2}\right)=\frac{31}{2}\times\frac{32}{2}=248$



A novel way to look at the average and sum of an A.P.

While all the questions of AP are based on the two formulae of T_n and S_n , one can reduce the amount of work in certain questions by using the understanding of the average of all terms of an AP...

Average of an AP:

The average of all terms of an AP is the middle term. It is also the average of the first and the last term. In fact, it is the average of any two terms equidistant from the center.

When the number of terms of an AP, say *n*, is odd, the middle term is given by $\frac{n+1}{2}^{ih}$ term and

when the number of terms is even i.e. when n is even, there would not be a specific middle term but there would be middle two terms. The average of these middle two terms will be the average of the AP.

E.g. 85: What is the average of all the terms of 101, 105, 109, 113, ..., 165, 169, 173.

The approach using formula would be as follows:

Finding the number of terms, $n = \frac{173 - 101}{4} + 1 = 19$

Sum of the terms = $\frac{19}{2} \times (2 \times 101 + 18 \times 4) = \frac{19}{2} \times (274)$

Average =
$$\frac{\text{Sum}}{\text{No. of terms}} = \frac{19 \times 274}{2 \times 19} = 137$$

Alternately, using the fact that average of all the terms of an AP is the average of the middle and the last term, we could have found the average much more easily as $\frac{101+173}{2} = \frac{274}{2} = 137$

Also note that the average of the second and the second last term; or the third and the third last term; and so on, will all be the same...

$$\frac{105+169}{2} = \frac{109+165}{2} = 137$$

E.g. 86: Find the sum of 25 terms of the AP: 63, 55, 47, 39, ...

The approach using formula would stay as done above i.e. finding the sum using the formula for S_{25} and then dividing this by 25 to find the average.

Alternately, since there are 25 terms, the middle term will be the 13th term and its value will be $63 + 12 \times (-8) = 63 - 96 = -33$. This will also be the average of all the terms as it is the middle term.



Exercise

1.	The sum of how many	terms of the AP –56, –4	9, -42,, would add up	42,, would add up to zero?				
	(1) 15	(2) 16	(3) 17	(4) 18				
2.	. Consider the AP: 168, 157, 146, 135, Another series is created by taking the 1 st , 4 th , 7 th , 10 th , terms of this A.P. What is the 10 th term in the new series thus formed?							
	(1) -75	(2) –129	(3)-140	(4) –151				
3.	The sum of the first 10 terms of an AP is equal to the sum of the next 5 terms of the AP. If the 15 th term of the series is 85, find the value of the common difference.							
	(1) 1	(2) 3	(3) 5	(4) 7				
4.	In an AP having 15 terms, the 8 th term is equal to 10. Find the sum of all the 15 terms of the series							
	(1) 150	(2) 80	(3) 120	(4) Cannot be determined				
5. If the sum of first 12 terms of an AP is equal to the sum of the first 18 terms of the same AP, find sum of the first 30 terms of the AP.								
	(1)-1	(2) 0	(3) 1	(4) Cannot be determined				
6.	. The sum of first 9 terms of an AP is 72 and the sum of the first 15 terms of the same AP is 150. Fin the sum of the first 21 terms of the same AP.							
	(1) 228	(2) 242	(3) 246	(4) 252				

7. Find the sum of the lengths all the vertical lines in the following figure. All the vertical lines are perpendicular to the base of the triangle. The base of the triangle is divided into 10 equal segments. The length of the leftmost vertical line is 15 cm as shown in the figure.



Base is divided into 10 equal segments

(1) 75

(2) 82.5

(3) 165

(4) Cannot be determined



Geometric Progression

An *A.P.* was got by adding a constant to the last term to get the successive terms. A *G.P.* is got by multiplying the last term by a constant to get the next term. The *G.P.* is also characterized by the first term, denoted by a and the constant with which each term is multiplied to get the next term. This constant is called common ratio and is denoted by r.

If
$$a = 2$$
 and $r = 3$, we get the *G.P.*: 2, 6, 18, 54, 162,
If $a = 32$ and $r = \frac{1}{2}$, we get the *G.P.*: 32, 16, 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$,

Just as in an *A*.*P*., we have formulae for finding the value of any n^{th} term and for finding the sum of first *n* terms of a *G*.*P*.

$$T_n = ar^{(n-1)}$$
 and $S_n = a \frac{r^n - 1}{r - 1}$

In the case of a *G.P.* there is a specific *G.P.* where *r* is less than 1 (to be accurate *r* has to be between -1 and 1) and the *G.P.* has infinite terms, e.g. 6, 2, $\frac{2}{3}$, $\frac{2}{9}$, $\frac{2}{27}$,

Since we are multiplying by r, which is less than 1, the terms of this G.P become smaller and smaller as the number of terms keeps increasing. Quite a few of the G.P questions are based on such a G.P. For such a decreasing G.P (r being less than 1), the sum of infinite terms is given

by
$$S_{\infty} = \frac{a}{1-r}$$

E.g. 87: If the first term of a *G.P.* is $\frac{1}{256}$ and the common ratio is -4, find the 10th term of the *G.P.*

$$T_{10} = \frac{1}{256} \times (-4)^{(10-1)} = \frac{1}{4^4} \times (-4)^9 = -4^5 = -1024$$

E.g. 88: What is the sum of the series 16, 8, 4, 2, 1, $\frac{1}{2}$, This is an infinite *G.P.* with *r* being less than 1 and hence we use the formula

$$S_{\infty} = \frac{a}{1-r}$$
 and get the sum of infinite terms $= \frac{16}{1-\frac{1}{2}} = 32$

E.g. 89: What is the sum of the series 16, -8, 4, -2, 1, $-\frac{1}{2}$,

This is an infinite G.P. and since with r is between -1 and 1, we can use the formula

$$S_{\infty} = \frac{a}{1-r}$$
 and get the sum of infinite terms $= \frac{16}{1-\left(-\frac{1}{2}\right)} = \frac{16}{\frac{32}{2}} = \frac{32}{3}$



E.g. 90: A rubber ball is thrown up and it reaches a height of 64 feets and then falls back. On each bounce it bounces back to $\frac{3^{th}}{4}$ of the earlier distance. Find the distance traveled

by the ball till it comes to rest.

The distances that the ball travels the first time and then on each bounce is 64,

$$64 \times \frac{3}{4} = 48$$
, $48 \times \frac{3}{4} = 36$, $36 \times \frac{3}{4} = 27$ and so on. Thus the distances traveled are in

a decreasing G.P. with first term being 64 and common ration being $\frac{3}{4}$.

The sum of the *G.P.* 64, 48, 36, 27, is
$$\frac{64}{1-\frac{3}{4}} = 256$$

Now the ball travels each of the distances 64, 48, 36, 27, twice, once while going up and once while falling down. Thus the total distance traveled by the ball till it comes to rest is $2 \times 256 = 512$.

Arithmetic Mean and Geometric Mean:

For any series of numbers $a_1, a_2, a_3, \ldots, a_n$, two means are defined as follows:

Arithmetic Mean, $AM = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$ and

Geometric Mean, GM = $\sqrt[n]{a_1 \times a_2 \times a_3 \times \dots \times a_n}$

While an AM and GM is defined for any series or collection of numbers (the series need not be an *A.P.* or *G.P.*), the AM of an *A.P.* and the GM of an *G.P.* has special significance.

The Arithmetic Mean of an Arithmetic Progression is the middle term.

The Arithmetic Mean of an Arithmetic Progression is also the average of the first and last term.

Thus, the AM of 4, 11, 18, 25, 32, 39, 46 is the middle term i.e. 25. Please note that 25 is also the average of the first and last term viz. 4 and 46.

If the number of terms in the *A*.*P*. is even, the AM is the average of the middle two terms or the average of the first and last term. Thus, one is saved the effort of adding all the terms and then dividing by the number of terms.

Similarly the Geometric Mean of a Geometric Progression is the middle term or else the Geometric Mean of the first and last term.

Thus the GM of 3⁵, 3⁶, 3⁷, 3⁸, 3⁹ is 3⁷. Also note that Geometric Mean of 3⁵ and 3⁹ is $\sqrt{3^5 \times 3^9} = \sqrt{3^{14}} = 3^7$



Exercise

8.	If in a <i>G.P.</i> , the sixth te	rm is $-\frac{1}{9}$ and the ninth	term is $\frac{1}{243}$, find the firm	st term?					
	(1) 81	(2) 27	(3) 9	(4) 3					
9.	Find the sum of the infinite G.P. 90, 60, 40,								
	(1) 180	(2) 225	(3) 270	(4) Infinite					
10.	A rubber ball is droppe	d from a height of 64 fee	ets. On each bounce it bo	bunces back to $\frac{3^{th}}{4}$ of the					
	earlier distance. Find th	ne distance traveled by the	ne ball till it comes to res	st.					
	(1) 192	(2) 256	(3) 448	(4) 512					
11.	E_1 is an equilateral triangle whose area is 16 sq. cm. The mid-points of the sides of the triangle are joined to form another triangle E_2 . The mid-points of the sides of E_2 are joined to form another triangle E_3 and the process is repeated infinite times. Find the sum of the areas of all triangles E_1 , E_2 , E_3 ,								
	(1) 64	(2) 32	(3) 16	(4) 64/3					
12.	The sum of infinite terr	ns of a geometric series	is $\frac{3}{2}$. The sum of the 1 st	$, 3^{rd}, 5^{th}, 7^{th}, \dots$ terms of the					
	same series is $\frac{9}{8}$. Find the common ratio of the G.P.								
	(1) 1	(2) 1/3	(3) 1 or 1/3	(4) Cannot be determined					
13.	Find the sum of 10 term	ns of $1 + 11 + 111 + 111$	1 +						
	$(1) \ \frac{10^{10} - 100}{81}$	$(2) \ \frac{10^{11} - 100}{81}$	$(3) \ \frac{10^{10} + 100}{81}$	$(4) \ \frac{10^{11} + 100}{81}$					



Arithmetico-Geometric Progression (AGP)

In this type of series, each term is obtained by multiplying a term of an AP with a term of an GP

e.g.
$$\frac{3}{5} + \frac{7}{5^2} + \frac{11}{5^3} + \frac{15}{5^4} + \dots$$
 or $2x$, $5x^2$, $8x^3$, $11x^4$, $14x^5$,

In the first series we see that the numerators are in an AP and the denominators form a G.P. Similarly in this second example we see that 2, 5, 8, 11, 14, ... is an AP and $x, x^2, x^3, x^4, x^5, ...$ is an GP.

Mostly in questions of this form, the GP is an infinite GP with |r| < 1 and the question requires you to find the sum of infinite terms of the AGP.

E.g. 91: Find the sum of infinite terms of the series, $\frac{3}{5} + \frac{7}{5^2} + \frac{11}{5^3} + \frac{15}{5^4} + \dots$

Let
$$S = \frac{3}{5} + \frac{7}{5^2} + \frac{11}{5^3} + \frac{15}{5^4} + \dots$$

$$\frac{S}{5} = \frac{3}{5^2} + \frac{7}{5^3} + \frac{11}{5^4} + \dots$$

$$S - \frac{S}{5} = \frac{3}{5} + \frac{7 - 3}{5^2} + \frac{11 - 7}{5^3} + \frac{15 - 11}{5^4} + \dots$$

$$\Rightarrow \frac{4}{5}S = \frac{3}{5} + \frac{4}{5^2} + \frac{4}{5^3} + \frac{4}{5^4} + \dots \Rightarrow \frac{4}{5}S = \frac{3}{5} + \frac{\frac{4}{5^2}}{1 - \frac{1}{5}} = \frac{3}{5} + \frac{1}{5} = \frac{4}{5} \Rightarrow S = 1$$

E.g. 92: Find the sum of infinite terms of the series, 2x, $5x^2$, $8x^3$, $11x^4$, $14x^5$,

Let
$$S = 2x + 5x^2 + 8x^3 + 11x^4 + 14x^5 + \dots$$
 (i)
 $x \times S = 2x^2 + 5x^3 + 8x^4 + 11x^5 + \dots$ (ii)
(i) - (ii), $S - xS = 2x + 3x^2 + 3x^3 + 3x^4 + 3x^5 + \dots$
 $S(1 - x) = 2x + \frac{3x^2}{1 - x} \Longrightarrow S = \frac{2x}{1 - x} + \frac{3x^2}{(1 - x)^2}$



User Defined Series

Each term of a series could also defined by the user and the series need not be AP, GP or AGP.

E.g. 93: Find the sum of first 10 terms of a series where the n^{th} term of the series is given by the formula $n \times (n + 4)$

The series can be generated by taking n as 1, 2, 3, ...

Thus the series is 5, 12, 21, 32, ...

To find the sum of such series, we need to make use of Sigma i.e. \sum

 \sum_{1}^{n} refers to the summation of all terms with values of the variable successively assuming 1, 2, 3, ..., *n*.

Thus, $\sum_{i=1}^{n} i$ refers to 1 + 2 + 3 + ... + n which we all know to be $\frac{n \times (n+1)}{2}$. This in short is

written as $\sum n = \frac{n(n+1)}{2}$

Similarly, $\sum_{i=1}^{n} i^2$ refers to $1^2 + 2^2 + 3^2 + ... + n^2$ and this is $\sum n^2 = \frac{n(n+1)(2n+1)}{6}$

And $\sum_{i=1}^{n} i^3$ refers to $1^3 + 2^3 + 3^3 + \dots + n^3$ and this is $\sum n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Using this notation we need to find $\sum_{n=1}^{10} n \times (n+4) = \sum_{n=1}^{10} n^2 + 4 \times \sum_{n=1}^{10} n^2$

$$=\frac{10\times11\times21}{6}+4\times\frac{10\times11}{2}=385+220=605$$

E.g. 94: Find the 25th term of the series and also the sum of first 40 terms of the series: 1, 3, 6, 10, 15, 21,

The difference between the successive terms of the series is the set of natural numbers...

Thus the second term = 1 + 2

The third term = 3 + 3 = 1 + 2 + 3



Similarly the fourth term = 6 + 4 = 1 + 2 + 3 + 4 and so on.

Thus it is the series where the n^{th} term is given by the sum of first *n* natural numbers. It would pay off very well if one remembers the series (1, 3, 6, 10...) and identifies its presence in any questions immediately.

Thus, the 25th term = $\sum_{25} n = \frac{25 \times 26}{2} = 325$

To find the sum of 25 terms we would have to do $\sum_{n=1}^{25} \left(\sum n \right) = \sum_{n=1}^{25} \frac{n \times (n+1)}{2}$

$$=\frac{1}{2}\sum_{25}n^{2}+\frac{1}{2}\sum_{25}n=\frac{1}{2}\times\frac{25\times26\times51}{6}+\frac{1}{2}\times\frac{25\times26}{2}=2925$$

E.g. 95: In a series, the sum of *n* terms of the series is given by $n \times (2n+1) \times (3n+2)$. Find the 8th term of the series.

In this question the expression for the sum of *n* terms of the series is given (rather than a term of the series) and we have to identify the series or a particular term of the series. In such a question, it pays to note that the *n*th term can be found using $t_n = S_n - S_{n-1}$, where S_n refers to the sum of *n* terms and similarly S_{n-1} refers to sum of (n-1) terms.

Thus, $t_8 = 8 \times 17 \times 26 - 7 \times 15 \times 23 = 3536 - 2415 = 1121$.

Series formed by an Iterative Relation

The user could also define a series by specifying the relation between successive terms of a series...

A series may be defined as: $t_n = 3 \times t_{n-1} - t_{n-2}$ for all natural numbers, *n*, greater than 2 and $t_1 = 1$, $t_2 = 1$.

Here the relation, $t_n = 3 \times t_{n-1} - t_{n-2}$, should be read as "any term = thrice the previous term – the vet previous term".

One should note that each of the following refer to the same relation as the above. It's just stated differently:

 $t_{n+1} = 3 \times t_n - t_{n-1}$ for all natural numbers, *n*, greater than 1

 $t_{n+2} = 3 \times t_{n+1} - t_n$ for all natural numbers, *n*

or for that matter even $t_{n+5} = 3 \times t_{n+4} - t_{n+3}$ for all integral values of *n* greater than -3.

The subscripts are used to just define the relative position.



Using this relation and since the first two terms are given, the series can be constructed fast enough starting form the beginning e.g. this series will be (the following is given the step wise construction of the series identifying one term at a time)...

- 1, 1, 2
- 1, 1, 2, 1
- 1, 1, 2, 1, 5
- 1, 1, 2, 1, 5, -2

E.g. 96: If $t_n = \frac{t_{n-1} + t_{n+1}}{2}$ for all natural numbers, *n*, greater than 1 and if $t_1 = 1$, $t_2 = 3$, find the

value of t_{100} .

We see that any term is the average of the previous and the next term.

Thus, 3 is the average of 1 and the next term. So the next term is 5.

Using the same logic, we can say that 5 is the average of the previous term, 3, and the next term.

Thus the next term is 7.

So the series can be constructed as 1, 3, 5, 7, 9,

The 100th term in this series is the 100th odd number i.e. 99.

Yet another type of series and a different approach to tackle is as explained in the example below...

E.g. 97: Find the sum: $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{90}$

The way to solve this is very innovative and it is unlikely that it would strike someone unless s/he is aware of it. So learn the approach well and see if you can apply the approach to questions that cannot be solved otherwise.

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \dots + \frac{1}{90} = \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots + \frac{1}{9 \times 10}$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{9} - \frac{1}{10}\right) = 1 - \frac{1}{10} = \frac{9}{10}$$



Exercise

14.	In a series, the sum of <i>n</i> terms of the series is given by $n \times (2n + 1) \times (3n + 2)$. Find the 8 th term of the series.								
	(1) 1121	(2) 1423	(3) 1234	(4) 1324					
15.	Find the sum of infinite	e series: $\frac{3}{4} + \frac{5}{36} + \frac{7}{324} + \frac{7}{324}$	$\frac{9}{2916}$ +						
	(1) 13/16	(2) 97/118	(3) 117/128	(4) 57/63					
16.	Find the sum of infinite	e series: $1 + \frac{4}{7} + \frac{9}{7^2} + \frac{16}{7^3}$	+						
	(1) 49/27	(2) 9/7	(3) 49/36	(4) 7/6					
17.	If $t_{n+2} = t_{n+1} - t_n$ for n b	eing any natural number	r and if $t_1 = 10.5, t_2 = 8.5$	5 find the value of t_{100} .					
	(1) 10.5	(2) 8.5	(3) 2	(4) -10.5					
18.	If $t_n = t_{n-1} - t_{n-2}$ for <i>n</i> b	eing any natural number	r greater than 2 and if t_1 :	= 10.5, $t_2 = 8.5$ find the value					
	of $t_1 + t_2 + t_3 + \dots + t_n$	t_{100} .							
	(1) 0	(2) -6.5	(3) 6.5	(4) 21					
19.	If $t_n = \frac{t_{n-1} + t_{n+1}}{2}$ for <i>n</i> b	being any natural number	greater than 1 and if $t_1 =$	$t_1, t_2 = 3$, find the value of t_{100} .					
	(1) 99	(2) 101	(3) 199	(4) 201					
20.	Find the sum: $\frac{3}{4} + \frac{5}{36} + \frac{5}{36}$	$-\frac{7}{144} + \frac{9}{400} + \dots + \frac{19}{8100}$	-						
	(1) 9/10	(2) 99/100	(3) 999/1000	(4) 19/20					
21.	Find the sum: $\frac{1}{3 \times 7} + \frac{1}{7}$	$\frac{1}{7 \times 11} + \frac{1}{11 \times 15} + \dots + \frac{1}{35}$	$\frac{1}{5\times 39}$						
	(1) 1/13	(2) 2/13	(3) 3/13	(4) 4/13					
22.	Find the sum: $\frac{1}{1 \times 3 \times 5}$	$+\frac{1}{3\times5\times7}+\frac{1}{5\times7\times9}+\dots$	$\dots + \frac{1}{95 \times 97 \times 99}$						
	(1) 3200/9603	(2) 1600/9603	(3) 1200/9603	(4) 800/9603					
23.	A series of terms t_1, t_2, t_3	t_3, \ldots are defined such the	hat $t_1 = 1, t_2 = 2 + 3, t_3 = 4$	$4 + 5 + 6, t_4 = 7 + 8 + 9 + 10$					
	and so on. Find the value	ue of t_{10} .							
	(1) 671	(2) 505	(3) 369	(4) 123					
24.	Find the 25 th term of th	e series: 1, 3, 6, 10, 15, 2	21,						
	(1) 300	(2) 325	(3) 351	(4) 377					
25.	Find the sum of first 10) terms of a series where	the n^{th} term of the series	s is given by the formula $n \times n$					
	(n+4)			(

(1) 505(2) 405(3) 605(4) 705



Answer Key

Basics of Algebra & Identities				Functions									
1.1	2.2	3.2	4.4	5.3	6.2	7.4	1.3	2.1	3.2	4.4	5.3	6. 1	7.1
8.3	9.4	10.4					8.3	9.2	10.3	11.2	12.3	13.4	14. 3
Factoris	ation						15.1						
1. i. $(x + 1)(x + 4)(x - 3)$ 1. ii. $(x - 2)(x + 3)(x - 3)$					Progress	sions							
1.iii. (<i>x</i> -	(+ 1)(2x -	1)(3x + 1))				1.3	2.2	3.3	4.1	5.2	6.4	7.2
2.3	3.1	4.4	5.2	6.3	7.3	8.3	8.2	9.3	10.3	11.4	12.2	13.2	14. 1
9.i. 1	9.ii. 4	9.iii. 2	9.iv. 3	10.i. 2	10.ii. 1		15.3	16.1	17.4	18.3	19.3	20.2	21.1
Equation	ns						22.4	23.2	24. 2	25.3			
1.i. 4	1.ii. 4	1.iii. 4	1.iv. 1	2.3	3.3	4.2							
5.2	6.3	7.1	8.3	9.i. 3	9.ii. 2	9.iii. 3							
10.2	11.2	12.3	13.3	14.2	15.1	16.4							
17.i. 2	17.ii. 1	17.iii. 3	18.2	19.4	20.i. 4	20.ii. 4							
21.4	22.3	23.3	24. 1										
Inequali	ities												
1.2	2.3	3.4	4.3	5.4	6.4	7.3							
8.3	9.2	10.1	11.3	12.3	13.1	14.1							
15.1	16.4	17.3	18.3	19.2	20. 1								
Modulu	S												
1.1	2.2	3.2	4.3	5.4	6.3	7.2							
8.4	9.4	10.4	11.2	12.1	13.3	14.2							
15.3													
Maxima	a Minima												
1.4	2.3	3.3	4.4	5.1	6.4	7.3							
8.4	9.4	10.2	11.2	12.2	13.4	14.1							
15.3	16.3	17.2	18.3	19.4	20.1	21.3							
22.3	23.4	24.4	25.2										



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